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小区间中的全平方数*

贾 朝 华
(北京大学数学系)

§1 引 言

一个正整数 n 是全平方数是指：p 是 n 的素因子，则 $p^2|n$. $Q(x)$ 表示不超过 x 的全体全平方数的个数. Bateman 和 Grosswald [1] 中证明了：

$$Q(x)=\frac{\zeta\left(\frac{3}{2}\right)}{\zeta(3)}x^{1/2}+\frac{\zeta\left(\frac{2}{3}\right)}{\zeta(2)}x^{1/3}+o(x^{1/6}).$$

P. Shiu [2] 中证明了小区间的结果：

设
$$h=x^{1/2+\theta}, \tag{1}$$
其中 $0.1526\leqslant\theta\leqslant\frac{1}{6}$, 则有

$$Q(x+h)-Q(x)\sim\frac{1}{2}\frac{\zeta\left(\frac{3}{2}\right)}{\zeta(3)}x^{\theta}, \tag{2}$$

这里 $f(x)\sim g(x)$ 表示 $\frac{f(x)}{g(x)}\to 1$ $(x\to\infty$ 时).

本文的目的是用较浅显的三角和估计改进[2]的结果，且指出进一步研究的方向，可以得出更好的结果，同时对熟悉各种三角和估计的应用是有帮助的.

定理 当 $0.1490342\leqslant\theta\leqslant\frac{1}{6}$ 时，(2)式成立.

我们以下设 $C, C_1, C_2, \cdots, A_1, A_2, \cdots$ 为正常数，在不同式子里有不同的意义.

§2 问题的转化

由[2]中的(3.4)式，我们有

$$\begin{aligned}Q(x+h)-Q(x)&=\sum_{x<a^2b^3m^6\leqslant x+h}\mu(m)\\&=\sum\nolimits_1+\sum\nolimits_2+\sum\nolimits_3\end{aligned} \tag{3}$$

其中

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$$\sum\nolimits_1 = \sum_{m\leqslant T_1}\mu(m)\left\{S\left(\frac{x+h}{m^6}\right)-S\left(\frac{x}{m^6}\right)\right\},$$

$$\sum\nolimits_2 = \sum_{T_1<m\leqslant T_2}\mu(m)\left\{S\left(\frac{x+h}{m^6}\right)-S\left(\frac{x}{m^6}\right)\right\},$$

$$\sum\nolimits_3 = \sum_{\substack{x<a^2b^3m^6\leqslant x+h\\ m>T_2}}\mu(m),$$

这里

$$S(x) = \sum_{a^2b^3\leqslant x} 1.$$

我们令

$$T_1 = x^{0.082337},\quad T_2 = x^{0.1225855}, \tag{4}$$

且设

$$0.1490342 < \theta < \frac{1}{6}. \tag{5}$$

又由[2]中(2.2)、(2.5)式可知

$$S(x) = \zeta\left(\frac{3}{2}\right)x^{1/2} + \zeta\left(\frac{2}{3}\right)x^{1/3} + \Delta(x), \tag{6}$$

其中

$$\Delta(x) = -\sum_{n\leqslant x^{1/5}}\left(\left(\frac{x^{1/2}}{n^{3/2}}\right)\right) - \sum_{n\leqslant x^{1/5}}\left(\left(\frac{x^{1/3}}{n^{2/3}}\right)\right) + O(1), \tag{7}$$

而

$$\Delta(x) \ll x^{\rho},\quad \rho = 0.1318162, \tag{8}$$

$$((x)) = \{x\} - \frac{1}{2}.$$

在 $\sum_1$ 中用(6)、(8)可得

$$\begin{aligned}\sum\nolimits_1 &= \frac{1}{2}\frac{\zeta\left(\frac{3}{2}\right)}{\zeta(3)}x^{\theta} + o(x^{\theta}) + O\left(\sum_{m\leqslant T_1}\left(\left|\Delta\left(\frac{x+h}{m^6}\right)\right| + \left|\Delta\left(\frac{x}{m^6}\right)\right|\right)\right)\\ &= \frac{1}{2}\frac{\zeta\left(\frac{3}{2}\right)}{\zeta(3)}x^{\theta} + o(x^{\theta}) + O\left(\sum_{m\leqslant T_1}\left(\frac{x}{m^6}\right)^{\rho}\right)\\ &= \frac{1}{2}\frac{\zeta\left(\frac{3}{2}\right)}{\zeta(3)}x^{\theta} + o(x^{\theta}).\end{aligned} \tag{9}$$

又有

$$\left|\sum\nolimits_2\right| \leqslant \sum_{T_1<m\leqslant T_2}\left(S\left(\frac{x+h}{m^6}\right) - S\left(\frac{x}{m^6}\right)\right).$$

由(6)可得

$$\left|\sum\nolimits_2\right| \leqslant o(x^{\theta}) + \sum_{T_1<m\leqslant T_2}\left(\Delta\left(\frac{x+h}{m^6}\right) - \Delta\left(\frac{x}{m^6}\right)\right). \tag{10}$$

我们只须估 $\sum\limits_{T_1<m\leqslant T_2}\Delta\left(\frac{x}{m^6}\right)$，而 $\sum\limits_{T_1<m\leqslant T_2}\Delta\left(\frac{x+h}{m^6}\right)$ 的估计同理可得，由(7)可得

$$\sum_{T_1<m\leqslant T_2}\Delta\left(\frac{x}{m^6}\right)$$

$$=\sum_{T_1<m\leqslant T_2}\sum_{n\leqslant\left(\frac{x}{m^6}\right)^{1/5}}\left(\left(\frac{x^{1/2}}{m^3n^{3/2}}\right)\right)+\sum_{T_1<m\leqslant T_2}\sum_{n\leqslant\left(\frac{x}{m^6}\right)^{1/5}}\left(\left(\frac{x^{1/3}}{m^2n^{2/3}}\right)\right)+o(x^\theta), \qquad (11)$$

这留待 § 3 中估计.

以下来看 $\sum_3$,

$$|\textstyle\sum_3|\leqslant T(x+h,\ T_2)-T(x,\ T_2), \qquad (12)$$

其中

$$T(x,\ T_2)=\sum_{\substack{a^2b^3m^6\leqslant x\\ m>T_2}}1.$$

我们设 $\tau(n)=\sum\limits_{a^2b^3=n}1$，且设 T_2 为半整数，从[2]中(3.10)及以下两个未标号的式子中，我们可得

$$T(x,\ T_2)=\sum_{n\leqslant\frac{x}{T_2^6}}\tau(n)\left(\frac{x}{n}\right)^{1/6}-\sum_{n\leqslant\frac{x}{T_2^6}}\tau(n)\left(\left(\frac{x^{1/6}}{n^{1/6}}\right)\right)-T_2S\left(\frac{x}{T_2^6}\right)$$

$$=\frac{3}{2}\zeta\left(\frac{3}{2}\right)\frac{x^{1/2}}{T_2^2}+2\zeta\left(\frac{2}{3}\right)\left(\frac{x^{1/3}}{T_2}\right)+\zeta\left(\frac{1}{3}\right)\zeta\left(\frac{1}{2}\right)x^{1/6}+T_2\Delta\left(\frac{x}{T_2^6}\right)$$

$$-\zeta\left(\frac{3}{2}\right)\frac{x^{1/2}}{T_2^2}-\zeta\left(\frac{2}{3}\right)\frac{x^{1/3}}{T_2}-T_2\Delta\left(\frac{x}{T_2^6}\right)+O\left(\sum_{n\leqslant\frac{x}{T_2^6}}\tau(n)\right)$$

$$=\frac{1}{2}\zeta\left(\frac{3}{2}\right)\frac{x^{1/2}}{T_2^2}+\zeta\left(\frac{2}{3}\right)\frac{x^{1/3}}{T_2}+\zeta\left(\frac{1}{3}\right)\zeta\left(\frac{1}{2}\right)x^{1/6}$$

$$+O\left(S\left(\frac{x}{T_2^6}\right)\right)T(x+h,\ T_2)-T(x,\ T_2)=o(x^\theta)+O\left(\frac{x^{1/2}}{T_2^3}\right)$$

$$=o(x^\theta),$$

代入(12)中，即知

$$\textstyle\sum_3=o(x^\theta). \qquad (13)$$

为了证明定理，以下我们只须估计(11)式.

§ 3 三角和估计

为估(11)式，我们只须估形如以下的和式

$$\textstyle\sum_A=\sum\limits_{M<m\leqslant 2M}\sum\limits_{N<n\leqslant 2N}\left(\left(\frac{x^{1/2}}{m^3n^{3/2}}\right)\right),$$

$$\Sigma_B = \sum_{M<m\leqslant 2M}\sum_{N<n\leqslant 2N}\left(\left(\frac{x^{1/3}}{m^2n^{2/3}}\right)\right),$$

其中 $N \leqslant \left(\frac{x}{M^6}\right)^{1/5}$，这样就有 $M \leqslant x^{1/6}$，$N \leqslant x^{1/5}$.

我们先来估 Σ_A，而对 Σ_B 的估计因步骤上有相似的地方，因而在证明时比较简略一些，用展开式

$$((\alpha)) = -\sum_{0<|h|\leqslant H}\frac{e(h\alpha)}{2\pi ih} + O\left(\min\left(1,\ \frac{1}{H\|\alpha\|}\right)\right),$$

将 Σ_A 转化为三角和:

$$\Sigma_A = \sum_{M<m\leqslant 2M}\sum_{N<n\leqslant 2N}\left(-\sum_{0<|h|\leqslant H}\frac{e\left(\frac{hx^{1/2}}{m^3n^{3/2}}\right)}{2\pi ih} + O\left(\min\left(1,\ \frac{1}{H\left\|\frac{x^{1/2}}{m^3n^{3/2}}\right\|}\right)\right)\right)$$

$$= -\sum_{0<|h|\leqslant H}\frac{1}{2\pi ih}\sum_{M<m\leqslant 2M}\sum_{N<n\leqslant 2N}e\left(\frac{hx^{1/2}}{m^3n^{3/2}}\right)$$

$$+ O\left(\sum_{M<m\leqslant 2M}\sum_{N<n\leqslant 2N}\min\left(1,\ \frac{1}{H\left\|\frac{x^{1/2}}{m^3n^{3/2}}\right\|}\right)\right).$$

并利用展开式

$$\min\left(1, \frac{1}{H\|\alpha\|}\right) = \sum_{h=-\infty}^{\infty} a_h e(h\alpha),$$

其中

$$a_0 \ll \frac{\log H}{H},\ a_h \ll \min\left(\frac{1}{|h|},\ \frac{H}{h^2}\right)\ (h \neq 0).$$

我们可得

$$\Sigma_A \ll \frac{MN\log H}{H} + \sum_{h=1}^{\infty}\min\left(\frac{1}{h},\ \frac{H}{h^2}\right)\left|\sum_{M<m\leqslant 2M}\sum_{N<n\leqslant 2N}e\left(\frac{hx^{1/2}}{m^3n^{3/2}}\right)\right| \tag{14}$$

我们以下引用[3]中的二条定理

引理 1　如 $0 < \lambda_2 \leqslant f''(x) \leqslant c\lambda_2$，则

$$\sum_{a<n\leqslant b} e(f(n)) \ll (b-a)\lambda_2^{1/2} + \lambda_2^{-1/2},$$

见[3] p. 49 定理 5.1.

引理 2　$\frac{A_1}{R} \leqslant f''(x) \leqslant \frac{A_2}{R}$，$|f'''(x)| \leqslant \frac{A_3}{RU}$，$f'(u_\gamma) = \gamma$，$[\alpha, \beta]$ 是 $[a, b]$在变换 $y = f'(x)$ 下的映象，则

$$\sum_{a<n\leqslant b} e(f(n)) = e\left(\frac{1}{8}\right)\sum_{\alpha<\gamma\leqslant\beta}\frac{e(f(u_\gamma) - \gamma u_\gamma)}{|f''(u_\gamma)|^{1/2}}$$

$$+ O((b-a+R)U^{-1} + \log(\beta-\alpha+2) + \sqrt{R}),$$

见[3] p. 36 定理 2.2.

设 I 是 $[M, 2M]$ 的子区间. 由引理 2，我们得

$$\sum_{M<m\leqslant 2M}\sum_{N<n\leqslant 4N} e\left(\frac{hx^{1/2}}{m^3n^{3/2}}\right)$$

$$\ll \left|\sum_{M<m\leqslant 4M}\left(\sum_{\frac{c_1hx^{1/2}}{m^3N^{5/2}}<\gamma\leqslant\frac{c_2hx^{1/2}}{m^3N^{5/2}}}\frac{e\left(\frac{c_3\gamma^{3/5}(hx^{1/2})^{2/5}}{m^{6/5}}\right)}{\sqrt{\frac{hx^{1/2}}{m^3u_\gamma^{7/2}}}}\right.\right.$$

$$\left.\left.+\log(hx)+\frac{M^3N^{5/2}}{hx^{1/2}}+\frac{M^{3/2}N^{7/4}}{(hx^{1/2})^{1/2}}\right)\right|$$

$$\ll \left|\sum_{\frac{c_1hx^{1/2}}{M^3N^{5/2}}<\gamma\leqslant\frac{c_2hx^{1/2}}{M^3N^{5/2}}}\left(\frac{M^{3/2}N^{7/4}}{h^{1/2}x^{1/4}}\right)\sum_{m\in I} e\left(\frac{c_3\gamma^{3/5}(hx^{1/2})^{2/5}}{m^{6/5}}\right)\right|$$

$$+M\log(hx)+\frac{M^4N^{5/2}}{hx^{1/2}}+\frac{M^{5/2}N^{7/4}}{(hx^{1/2})^{1/2}}$$

由引 1，

$$\sum_{M<m\leqslant 2M}\sum_{N<n\leqslant 4N} e\left(\frac{hx^{1/2}}{m^3n^{3/2}}\right)$$

$$\ll \left|\sum_{\frac{c_1hx^{1/2}}{M^3N^{5/2}}<\gamma\leqslant\frac{c_2hx^{1/2}}{M^3N^{5/2}}}\left(\frac{M^{3/2}N^{7/4}}{h^{1/2}x^{1/4}}\right)\left(M\cdot\left(\frac{\gamma^{3/5}(hx^{1/2})^{2/5}}{M^{6/5+2}}\right)^{1/2}\right.\right.$$

$$\left.\left.+\left(\frac{M^{6/5+2}}{\gamma^{3/5}(hx^{1/2})^{2/5}}\right)^{1/2}\right)\right|+M\log(hx)+\frac{M^4N^{5/2}}{hx^{1/2}}+\frac{M^{5/2}N^{7/4}}{h^{1/2}x^{1/4}}$$

$$\ll \frac{M^{3/2}N^{7/4}}{h^{1/2}x^{1/4}}\left(M\cdot\frac{(hx^{1/2})^{1/5}}{M^{3/5+1}}\left(\frac{hx^{1/2}}{M^3N^{5/2}}\right)^{13/10}+\frac{M^{1+3/5}}{(hx^{1/2})^{1/5}}\left(\frac{hx^{1/2}}{M^3N^{5/2}}\right)^{7/10}\right)$$

$$+M\log(hx)+\frac{M^4N^{5/2}}{hx^{1/2}}+\frac{M^{5/2}N^{7/4}}{h^{1/2}x^{1/4}}$$

$$\ll \frac{hx^{1/2}}{M^3N^{3/4}}+M\log(hx)+\frac{M^4N^{5/2}}{hx^{1/2}}+\frac{M^{5/2}N^{7/4}}{h^{1/2}x^{1/4}}. \tag{15}$$

由(14)、(15)式

$$\sum\nolimits_{\Delta} \ll \frac{MN\log H}{H}+\sum_{h\leqslant HMN}\min\left(\frac{1}{h},\frac{H}{h^2}\right)\left|\sum_{M<m\leqslant 2M}\sum_{N<n\leqslant 2N} e\left(\frac{hx^{1/2}}{m^3n^{3/2}}\right)\right|$$

$$+\sum_{h>HMN}\min\left(\frac{1}{h},\frac{H}{h^2}\right)\left|\sum_{M<m\leqslant 2M}\sum_{N<n\leqslant 2N} e\left(\frac{hx^{1/2}}{m^3n^{3/2}}\right)\right|$$

$$\ll \frac{MN\log H}{H}+\frac{Hx^{1/2}}{M^3N^{3/4}}\log(Hx)+M\log^3(Hx)+\frac{M^4N^{5/2}}{x^{1/2}}$$

$$+\frac{M^{5/2}N^{7/4}}{x^{1/4}}+O(1).$$

在最后一个和式中，我们用了估计

$$\sum_{M<m\leqslant 2M}\ \sum_{N<n\leqslant 2N} e\left(\frac{hx^{1/2}}{m^3n^{3/2}}\right) \ll MN.$$

我们取 $H=\frac{M^2N^{5/4}}{x^{1/4}}$，则 $H\ll x^3$，有

$$\sum_A \ll \frac{x^{1/4}}{MN^{1/4}}\log x + M\log x + \frac{M^4N^{5/2}}{x^{1/2}} + \frac{M^{5/2}N^{7/4}}{x^{1/4}}.$$

我们用 $N\leqslant\left(\frac{x}{M^6}\right)^{1/5}$ 代入上式，且注意 $M\leqslant x^{1/6}$，则

$$\begin{aligned}\sum_A &\ll \frac{x^{1/4}}{MN^{1/4}}\log x + M\log x + x^{1/10}M^{2/5}\\ &\ll \left(\frac{x^{1/4}}{MN^{1/4}} + x^{1/10}M^{2/5}\right)\log x.\end{aligned} \tag{16}$$

我们再来用引 1 估，

$$\begin{aligned}&\sum_{M<m\leqslant 2M}\ \sum_{N<n\leqslant 2N} e\left(\frac{hx^{1/2}}{m^3n^{3/2}}\right)\\ &\quad=\sum_{N<n\leqslant 2N}\left(\sum_{M<m\leqslant 2M} e\left(\frac{hx^{1/2}}{m^3n^{3/2}}\right)\right)\\ &\quad\ll \sum_{N<n\leqslant 2N}\left(M\left(\frac{hx^{1/2}}{M^5N^{3/2}}\right)^{1/2} + \left(\frac{M^5N^{3/2}}{hx^{1/2}}\right)^{1/2}\right)\\ &\quad\ll (hx^{1/2})^{1/2}N^{1/4}M^{-3/2} + M^{5/2}N^{7/4}(hx^{1/2})^{-1/2}.\end{aligned}$$

取 $H=M^{5/3}N^{1/2}x^{-1/6}$，

$$\begin{aligned}\sum_A &\ll \frac{MN}{H}\log H + \frac{H^{1/2}x^{1/4}N^{1/4}}{M^{3/2}} + \frac{M^{5/2}N^{7/4}}{x^{1/4}}\\ &\ll x^{1/6}N^{1/2}M^{-2/3}\log x + x^{1/10}M^{2/5}\log x.\end{aligned} \tag{17}$$

由(16)、(17)可得

$$\begin{aligned}\frac{1}{\log x}\sum_A &\ll \min\left(\frac{x^{1/4}}{MN^{1/4}} + x^{1/10}M^{2/5},\ x^{1/6}N^{1/2}M^{-2/3} + x^{1/10}M^{2/5}\right)\\ &\ll \min\left(\frac{x^{1/4}}{MN^{1/4}},\ x^{1/6}N^{1/2}M^{-2/3}\right) + x^{1/10}M^{2/5}\\ &\ll (x^{1/6}N^{1/2}M^{-2/3})^{1/3}\left(\frac{x^{1/4}}{MN^{1/4}}\right)^{2/3} + x^{1/10}M^{2/5}\\ &= x^{2/9}M^{-8/9} + x^{1/10}M^{2/5},\end{aligned}$$

所以，

$$\sum_A \ll (x^{2/9}M^{-8/9} + x^{1/10}M^{2/5})\log x. \tag{18}$$

我们以下来估 $\sum_B$.

$$\sum_B \ll \frac{MN\log H}{H} + \sum_{n=1}^{\infty}\min\left(\frac{1}{h},\ \frac{H}{h^2}\right)\left|\sum_{M<m\leqslant 2M}\ \sum_{N<n\leqslant 2N} e\left(\frac{hx^{1/3}}{m^2n^{2/3}}\right)\right|$$

用引 2 知，

$$\sum_{M<m\leqslant 2M}\ \sum_{N<n\leqslant 2N} e\left(\frac{hx^{1/3}}{m^2n^{2/3}}\right)$$

$$\ll \sum_{M<m\leq 2M}\left(\sum_{\frac{c_1hx^{1/3}}{m^2N^{5/3}}<r\leq\frac{c_2hx^{1/3}}{m^2N^{5/3}}}\frac{e\left(\frac{c_3r^{2/5}(hx^{1/3})^{3/5}}{m^{6/5}}\right)}{\sqrt{\frac{hx^{1/3}}{m^2u_r^{8/3}}}}\right.$$

$$\left.+\log(hx)+\frac{M^2N^{5/3}}{hx^{1/3}}+\frac{MN^{4/3}}{h^{1/3}x^{1/6}}\right)$$

$$\ll \sum_{\frac{c_1hx^{1/3}}{M^2N^{5/3}}<r\leq\frac{c_2hx^{1/3}}{M^2N^{5/3}}}\left(\frac{MN^{4/3}}{h^{1/2}x^{1/6}}\right)\sum_{m\in I}e\left(\frac{c_3r^{2/5}(hx^{1/3})^{3/5}}{m^{6/5}}\right)$$

$$+M\log(hx)+\frac{M^3N^{5/3}}{hx^{1/3}}+\frac{M^2N^{4/3}}{h^{1/2}x^{1/6}},$$

并用引1可得

$$\sum_{M<m\leq 2M}\sum_{N<n\leq 2N}e\left(\frac{hx^{1/3}}{m^2n^{2/3}}\right)$$

$$\ll \sum_{\frac{c_1hx^{1/3}}{M^2N^{5/3}}<r\leq\frac{c_2hx^{1/3}}{M^2N^{5/3}}}\left(\frac{MN^{4/3}}{h^{1/2}x^{1/6}}\right)\left(M\cdot\left(\frac{r^{2/5}(hx^{1/3})^{3/5}}{M^{6/5+2}}\right)^{1/2}\right.$$

$$\left.+\left(\frac{M^{6/5+2}}{r^{2/5}(hx^{1/3})^{3/5}}\right)^{1/2}\right)+M\log(hx)+\frac{M^3N^{5/3}}{hx^{1/3}}+\frac{M^2N^{4/3}}{h^{1/2}x^{1/6}}$$

$$\ll\left(\frac{MN^{4/3}}{h^{1/2}x^{1/6}}\right)\left(M\cdot\frac{(hx^{1/3})^{3/10}}{M^{3/5+1}}\left(\frac{hx^{1/3}}{M^2N^{5/3}}\right)^{6/5}+\frac{M^{3/5+1}}{(hx^{1/3})^{3/10}}\left(\frac{hx^{1/3}}{M^2N^{5/3}}\right)^{4/5}\right)$$

$$+M\log(hx)+\frac{M^3N^{5/3}}{hx^{1/3}}+\frac{M^2N^{4/3}}{h^{1/2}x^{1/6}}$$

$$\ll\frac{hx^{1/3}}{M^2N^{2/3}}+M\log(hx)+\frac{M^3N^{5/3}}{hx^{1/3}}+\frac{M^2N^{4/3}}{h^{1/2}x^{1/6}}$$

令 $H=M^{3/2}N^{5/6}x^{-1/6}$，且由 $N\leq\left(\frac{x}{M^6}\right)^{1/5}$，可得

$$\sum\nolimits_B\ll\frac{MN}{H}\log H+\frac{Hx^{1/3}}{M^2N^{2/3}}\log(Hx)+M\log(Hx)+\frac{M^3N^{5/3}}{x^{1/3}}+\frac{M^2N^{4/3}}{x^{1/6}}$$

$$\ll\left(x^{1/6}N^{1/6}M^{-1/2}+M+\frac{M^3N^{5/3}}{x^{1/3}}+\frac{M^2N^{4/3}}{x^{1/6}}\right)\log x$$

$$\ll(x^{1/5}M^{-7/10}+M+x^{1/10}M^{2/5})\log x$$

$$\ll(x^{1/5}M^{-7/10}+x^{1/10}M^{2/5})\log x,\tag{19}$$

由(18)、(19)得

$$\sum\nolimits_A+\sum\nolimits_B\ll(x^{2/9}M^{-8/9}+x^{1/5}M^{-7/10}+x^{1/10}M^{2/5})\log x\tag{20}$$

§4 定理的证明及注

由(11)及(20)式可得

$$\sum_{T_1 < m \leqslant T_2} \Delta\left(\frac{x}{m^6}\right)$$
$$\ll (x^{2/9}T_1^{-8/9} + x^{1/5}T_1^{-7/10} + x^{1/10}T_2^{2/5}) \log x + o(x^\theta)$$
$$= o(x^\theta),$$

因而有

$$\sum\nolimits_2 = o(x^\theta). \tag{21}$$

把(3)、(9)、(13)、(21)合起来，可得定理.

注　我们在 § 3 中作三角和估计时，可用较深的三角和估计，如 Phillips [4]的指数偶理论、Srinivasan [5] 的二维指数偶理论，可得更好的结果，有兴趣的读者不妨一试，但要特别注意所涉及的定理中条件的验证.

参　考　文　献

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