# 四值命題演算與四色問題

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**摘要：** 本文指出四値命題演算與四色問題之間的一個關係：在四値命題演算中定義一個命題類；容易證明，如果這類中的每一分子都是四値命題演算中的定理，那麼四色問題即完全解決，這樣，就把四色問題還原至四値命題演算的問題了．本文最後說明本文對於四色問題雖無直接貢獻，然由於數理邏輯的成果，提供了一些研究四色問題新方向的可能性．

# A NOTE ON THE 4-VALUED PROPOSITIONAL CALCULUS AND THE FOUR COLOUR PROBLEM

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The aim of the present paper is to describe briefly a certain relationship between 4-valued propositional calculus and the four-colour problem. The relationship here stated will eventually prove itself to be easily understandable to the reader who has acquainted himself with the nature of the four-colour problem in combinatorial topology and have also some idea of mathematical logic, especially of the many-valued propositional calculus.

Now, we presume that the reader has already known the import of the Rosser and Turquette's artical[①] (RT) and we shall adopt the notations they worked with in their artical RT. Furthermore we shall make free use of all the conventions commonly observed by the mathematical logicians, especially we shall adopt the customary, self-explanatory, usage according

① J. B. Rosser and A. R. Turquette, Axiom schemes for $m$-valued propositional calculi, Journal of Symbolic Logic, vol. 10 (1945). pp. 61-82.

to which symbols belonging to the propositional calculus, and symbols introduced by formal definitions, are used in the intuitive discussions as names for themselves; '$\varepsilon$', '$\subseteq$', '$\cup$' '$\{x_1, \cdots, x_n\}$' are used with their usual set-theoretic meanings. Thus '$\{x_1, \cdots, x_n\}$' stands for the set of which $x_1, \cdots, x_n$ constitute the members.

Let $p_1, p_2, p_3, \cdots$ be the infinite sequence of propositional variables in their alphabatical order.

D (= *Definition*) 1. $x \,\varepsilon\, \mathbf{pr}$, *if and only if, $x$ is a propositional variable.*

D2. $\beta \,\varepsilon\, \mathbf{coup}(n)$, *if and only if, there are $x, y$ such that, $x \neq y$, $\beta = \{x, y\}$ and $x, y \,\varepsilon\, \{p_1, \cdots, p_n\}$.*

D3. $\zeta \,\varepsilon\, \mathbf{grc}(n)$, *if and only if, the following two conditions are satisfied:*

(1) $\zeta \subseteq \mathbf{coup}(n)$,

(2) *for any $m_1, m_2, m_3, m_4$, if $m_1 < m_2 < m_3 < m_4$, and if $\{p_{m_1}, p_{m_2}\} \,\varepsilon\, \zeta$, then not $\{p_{m_3}, p_{m_4}\} \,\varepsilon\, \zeta$.*

D4. $\zeta \,\varepsilon\, \mathbf{gr}(n)$, *if and only if, there are $m_1, m_2$ that satisfy the following three conditions:*

(1) $n = \mathrm{Max}(m_1, m_2)$,

(2) *there are $\zeta_1, \zeta_2$ such that $\zeta_1 \,\varepsilon\, \mathbf{grc}(m_1), \zeta_2 \,\varepsilon\, \mathbf{grc}(m_2)$ and $\zeta = \zeta_1 \cup \zeta_2$,*

(3) *$\zeta$ is not empty.*

**D5.** $E\mathrm{PQ} =_{df} KC\mathrm{PQ}C\mathrm{QP}$

**D6.** $J^*\mathrm{PQ} = NJ_1 E\mathrm{PQ}$

**D7.** $(m_1, m_2, m_3, m_4)\,\mathrm{P} =_{df} \sum_{i=1}^{4} KU_{m_i}(\mathrm{P})\, J_i\,(\mathrm{P})$[2]

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[2]
$$\sum_{m=1}^{1} \mathrm{P}_m =_{df} \mathrm{P}_1,$$

$$\sum_{m=1}^{n+1} \mathrm{P}_m =_{df} A \sum_{m=1}^{n} \mathrm{P}_m \mathrm{P}_{n+1}.$$

See RT p. 72.

D8. $B_1 \mathrm{P} =_{df} (1,2,3,4)\,\mathrm{P}$
$B_2 \mathrm{P} =_{df} (2,3,4,1)\,\mathrm{P}$
$B_3 \mathrm{P} =_{df} (3,4,1,2)\,\mathrm{P}$
$B_4 \mathrm{P} =_{df} (4,1,2,3)\,\mathrm{P}$

D9. $\mathrm{P}\,\varepsilon\,\mathbf{dif}(\zeta)$, *if and only if, there are* $m$, $n$ *such that* $m < n$, $\{p_m, p_n\}\,\varepsilon\,\zeta$ *and* $\mathrm{P} = J^* p_m p_n$.

D10. *Suppose that all the propositional variables occur in* P *belonging to the set* $\{p_1, \cdots, p_n\}$, *we put*

$$\exists^n \mathrm{P} =_{df} \sum_{m_1=1}^{4} \cdots \sum_{m_n=1}^{4} \mathrm{P}(B_{m_1} p_1, \cdots, B_{m_n} p_n),$$

where $\mathrm{P}(B_{m_1} p_1, \cdots, B_{m_n} p_n)$ *is the formula arising out of* P *by substituting* $B_{m_1} p_1, \cdots, B_{m_n} p_n$ *into* $p_1, \cdots, p_n$ *respectively and simultaneously.*

D11. $\zeta\,\varepsilon\,\mathbf{col}$, *if and only if, there is a function* $f$ *that satisfies the following three conditions:*

(1) $f$ *is defined on the set of all* $x$ *such that there is an* $\alpha$, $x\,\varepsilon\,\alpha\,\varepsilon\,\zeta$,

(2) *for all* $x$, *if* $x$ *belongs to the argument of* $f$, *then* $f(x)\,\varepsilon\,\{1,2,3,4\}$,

(3) *for all* $x$ *and* $y$, *if* $\{x,y\}\,\varepsilon\,\zeta$, *then* $f(x) \neq f(y)$.

THEOREM A. *Suppose*

(a) $\zeta\,\varepsilon\,\mathbf{gr}(n)$,

(b) $\mathbf{dif}(\zeta) = \{\mathrm{P}_1, \cdots, \mathrm{P}_k\}$,

*then the following two conditions,* (c) *and* (d) *are equivalent:*

(c) *the formula*

$$\exists^n \prod_{1}^{k} \mathrm{P}_i \quad \cdots\cdots\cdots\cdots \quad Q\text{[3]}$$

*is a formal theorem of a complete 4-valued propositionsl calculus, in other words, Q is derivable from* $(a_2), \cdots, (a_{14})$ *of* RT *for the case* $m = 4$;

[3] See RT p. 69.

(d) $\zeta \varepsilon$ **col**.

From the above THEOREM A. we can see that the relationship between the four-colour problem and the 4-valued propositional calculus has been established, for the four-colour problem may be taken as the proof of

($T_1$) (a) implies (d),

but the answer to the problem whether

($T_2$) ((a) and (b)) implies (c)

is a problem conserning the 4-valued propositional calculus. If ($T_2$) can be proved, then from THEOREM A. ($T_1$) follows, that is a complete solution of the four-colour problem is reached. Thus the four-colour problem is reduced to a problem of the 4-valued propositional calculus. The truth of THEOREM A. is evident and a formal proof of which is not difficult, so it is left here to the reader.

Though the relationship here described offers no direct contribution to the solution of the four-colour problem itself, it may have thrown some light upon the possibility of finding a new approach towards the solution with the help of the fruitful results in the mathematical logic.

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