THE GEOMETRY OF HIGHER PATH-SPACES

BY SHIING-SHEN CHERN (陳省身)

(Dedicated to Professor Elie Cartan on His Seventieth Birthday)

Introduction

The geometry in a space R_n $(x^1, \cdots, x^n)$ in which there is given a system of differential equations of the r-th order:

$$(1)\qquad \frac{d^r x^i}{dt^r} + F^i\left(\frac{d^{r-1}x}{dt^{r-1}}, \cdots, \frac{dx}{dt}, x, t\right) = 0,^{(1)} \qquad r \geqq 2,$$

has recently been studied by various writers.(2) It seems to me that the most natural way to study this problem is to formulate the problem as a problem of equivalence. By solving completely the problem of equivalence the geometry in the space is automatically defined. By the use of the general method of Cartan the problem of equivalence and the definition of a geometry from a geometric object become two aspects of the same problem. We want to illustrate this point clearly in the discussion of the present problem.

The paper is divided into five sections. In §1 we give the definition of a generalized geometry in the $(rn+1)$-dimensional space $\left(\frac{d^{r-1}x}{dt^{r-1}}, \frac{d^{r-2}x}{dt^{r-2}}, \cdots, x, t\right)$, the definition being invariant under the group of transformations on the variables (x^i, t):

$$(2)\qquad \begin{cases} \bar{x}^i = \bar{x}^i(x^1, \cdots, x^n, t), \\ \bar{t} = t \end{cases} \qquad \left|\frac{\partial \bar{x}^i}{\partial x^k}\right| \neq 0.$$

The discussion is based on the general theory of Cartan and we give in §2 the results in terms of the notations of the absolute differential

(1) We agree that every small Latin index takes the values from 1 to n and that repeated indices represent a sum of the terms.

(2) Cf. the bibliography at the end of the paper.

calculus. The calculations are carried out in the special cases $r=2$ and $r=3$ in §3. In §4 we study the geometry of the differential system (1) under the subgroup of the group of transformations (2) by which the $\bar{x}^i$ are functions of $x^1, \cdots, x^n$ alone. In §5 some particular cases are discussed.

§1. The Problem of Equivalence

Suppose there be given a system of differential equations in a space $\bar{R}_n$ $(\bar{x}^1, \cdots, \bar{x}^n)$ of the same form as (1):

$$\frac{d^r\bar{x}^i}{d\bar{t}^r} + \bar{F}^i\left(\frac{d^{r-1}\bar{x}}{d\bar{t}^{r-1}}, \cdots, \frac{d\bar{x}}{d\bar{t}}, \bar{x}, \bar{t}\right) = 0. \tag{1'}$$

We propose to find the conditions that there exists a transformation of the group (2) which carries the system (1′) into (1). For this purpose put

$$x_1^i = \frac{dx^i}{dt}, \cdots, x_{r-1}^i = \frac{d^{r-1}x^i}{dt^{r-1}}. \tag{3}$$

The system (1) is then equivalent to the Pfaffian system

$$\begin{cases} dx^i - x_1^i\, dt = 0, \\ dx_1^i - x_2^i\, dt = 0, \\ \cdots\cdots\cdots \\ dx_{r-2}^i - x_{r-1}^i\, dt = 0, \\ dx_{r-1}^i + F^i(x_{r-1}, \cdots, x_1, x, t)\, dt = 0. \end{cases} \tag{4}$$

Put again

$$\begin{cases} \omega_1^i = a_{1,k}^i\left(dx^k - x_1^k\, dt\right), \\ \omega_2^i = a_{2,k}^i\left\{dx_1^k - x_2^k\, dt + a_{2\,1,j}^k(dx^j - x_1^j\, dt)\right\}, \\ \cdots\cdots\cdots\cdots \\ \omega_s^i = a_{s,k}^i\left\{dx_{s-1}^k - x_s^k\, dt + a_{s\,1,j}^k(dx_{s-2}^j - x_{s-1}^j\, dt) + \cdots \right. \\ \qquad\qquad \left. + a_{s\,s-1,j}^k(dx^j - x_1^j\, dt)\right\}, \\ \cdots\cdots\cdots\cdots \end{cases} \tag{5}$$

$$\left\{ \begin{aligned} \omega_r^i = a_{r,k}^i \Big\{ & dx_{r-1}^k + F^k dt + a_{r\,1,\,j}^k (dx_{r-2}^j - x_{r-1}^j dt) + \cdots \\ & + a_{r\,r-1,\,j}^k (dx^j - x_1^j dt) \Big\} , \end{aligned} \right.$$

where the a's are auxiliary variables. We can define the $\bar{\omega}_1^i, \cdots, \bar{\omega}_r^i$ from the system (1′) in the same way as the ω's are defined from the system (1). *In order that the systems (1) and (1′) be equivalent under the group of transformations (2) it is necessary and sufficient that there be a transformation in the rn+1 variables*

$$x_{r-1}^i,\ x_{r-2}^i, \cdots, x^i, t.$$

and in the auxiliary variables

$$a_{1,\,k}^i,\ a_{2,\,k}^i,\ \cdots,\ a_{r\,r-1,\,j}^k$$

such that the following equations hold:

$$\bar{\omega}_1^i = \omega_1^i, \ \cdots, \ \bar{\omega}_r^i = \omega_r^i. \tag{6}$$

To solve this problem of equivalence we try to reduce the number of auxiliary variables(3) by imposing on the ω's conditions of intrinsic nature. The relations

$$d\omega_s^i \equiv -\,\omega_{s+1}^i\,dt \qquad (\text{mod. } \omega_1, \cdots, \omega_s) \tag{7}$$

$$s = 1, 2, \cdots, r-1$$

give

$$a_{s+1,\,k}^i = a_{s,\,k}^i$$

or

$$a_{r,\,k}^i = a_{r-1,\,k}^i = \cdots = a_{1,\,k}^i = a_k^i \text{ (say)}. \tag{8}$$

(3) For the meaning of the so-called exterior differential forms and exterior derivation cf. E. Cartan, *Leçons sur les invariants intégraux*, Chap. VI, VII, Paris 1922 and E. Kähler, *Einführung in die Theorie der Systeme von Differentialgleichungen*, Hamburg 1934. We have here employed the notation of Kähler.

From (7) with $s=1$ we may put

$$d\omega_1^i = \theta_j^i \omega_1^j - \omega_2^j \, dt, \tag{9}$$

where θ_j^i are Pfaffian forms in $x_{r-1}^i, \cdots, x^i, t, a_k^i, a_{21,j}^k$. On defining A_i^k by the relations

$$a_i^j A_j^k = a_j^k A_i^j = \delta_i^k \tag{10}$$

we have

$$\theta_j^i \equiv da_k^i A_j^k - a_k^i a_{21,l}^k A_j^l \, dt \qquad (\text{mod. } \omega_1) \tag{11}$$

Of course, by the form of the equations (9) the θ_j^i are not uniquely determined.

From (5) we have

$$d\omega_s^i \equiv da_k^i A_j^k \omega_s^j - a_k^i dx_s^k \, dt - a_k^i a_{s1,j}^k dx_{s-1}^j \, dt$$

$$(\text{mod. } \omega_1, \cdots, \omega_{s-1}),$$
$$s = 2, 3, \cdots, r-1.$$

The conditions

$$d\omega_s^i \equiv \theta_j^i \omega_s^j - \omega_{s+1}^i \, dt \qquad (\text{mod. } \omega_1, \cdots, \omega_{s-1}). \tag{12}$$

then give

$$a_{s+1\,1,j}^k = a_{s\,1,j}^k + a_{2\,1,j}^k$$

or

$$a_{s\,1,j}^k = (s-1) a_{2\,1,j}^k, \qquad s = 2, 3, \cdots, r. \tag{13}$$

From the last equation in (5) we get

$$d\omega_r^i \equiv da_k^i A_j^k \omega_r^j + a_k^i \left\{ dF^k dt - a_{r1,j}^k dx_{r-1}^j dt \right\},$$

$$(\text{mod. } \omega_1, \cdots, \omega_{r-1}).$$

But

$$dF^k\,dt \equiv \frac{\partial F^k}{\partial x^j_{r-1}}\,dx^j_{r-1}\,dt \qquad (\text{mod. } \omega_1, \cdots, \omega_{r-1}).$$

Therefore from (11), the condition

$$d\omega^i_r \equiv \theta^i_j\,\omega^j_r \qquad (\text{mod. } \omega_1, \cdots, \omega_{r-1}). \tag{14}$$

implies

$$a^k_{21,j} = \frac{1}{r}\,\frac{\partial F^k}{\partial x^j_{r-1}}. \tag{15}$$

From (15) and (13) the $a^k_{21,j}, \cdots, a^k_{r1,j}$ are all determined.

We proceed to determine the auxiliary variables $a^k_{sq,j}$ for which $2 \leq q \leq s-1$. *We shall show by mathematical induction that it is possible to determine these variables $a^k_{sq,j}$ by imposing on the ω's some conditions of intrinsic nature.* Suppose of the quantities

$$a^k_{\alpha\beta,j}, \; 1 \leq \beta \leq \alpha-1, \; \alpha = 2, 3, \cdots, r \tag{16}$$

all those for which

$$2 \leq \alpha \leq s-1$$

and those for which

$$\alpha = s, \quad 1 \leq \beta \leq q-1$$

are determined to be functions of the $rn+1$ variables

$$x^i_{r-1}, \; x^i_{r-2}, \; \cdots, \; x^i_1, \; x^i, \; t. \tag{17}$$

To determine the $a^k_{sq,j}$ also as functions of the variables (17) we seek for the conditions expressing that the exterior derivatives $d\omega^i_{s-1}$ do not contain the terms $\omega^k_{s-q}\,dt$ and show that these conditions suffice to determine $a^k_{sq,j}$.

In calculating the coefficient of $\omega^k_{s-q}\,dt$ in $d\omega^i_{s-1}$ we may neglect the terms in $\omega_1, \cdots, \omega_{s-q-1}$. We may thus use congruences mod. $\omega_1, \cdots, \omega_{s-q-1}$ in our proof. This gives in particular

$$dx^i \equiv x_1^i \, dt, \; \cdots, \; dx_{s-q-2}^i \equiv x_{s-q-1}^i \, dt. \tag{18}$$

Now, from (5),

$$\omega_{s-1}^i = a_k^i \Big\{ dx_{s-2}^k - x_{s-1}^k \, dt + a_{s-1\,1,\,j}^k \, (dx_{s-3}^j - x_{s-2}^j \, dt) + \cdots$$
$$+ a_{s-1\,s-2,\,j}^k \, (dx^j - x_1^j \, dt) \Big\},$$

in which the $a_{s-1\,1,\,j}^k, \cdots, a_{s-1\,s-2,\,j}^k$ are by hypothesis known functions of the variables in (17). Therefore we have

$$d\omega_{s-1}^i \equiv da_k^i \, A_j^k \, \omega_{s-1}^j + a_k^i \Big\{ - dx_{s-1}^k \, dt - a_{s-1\,1,\,j}^k \, dx_{s-2}^j \, dt - \cdots$$
$$- a_{s-1\,q,\,j}^k \, dx_{s-q-1}^j \, dt + da_{s-1\,1,\,j}^k \, (dx_{s-3}^j - x_{s-2}^j \, dt) + \cdots$$
$$+ da_{s-1\,q-1,\,j}^k \, (dx_{s-q-1}^j - x_{s-q}^j \, dt) \Big\}.$$

By introducing the notation

$$\frac{d\Phi}{dt} = \frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial x^i} x_1^i + \cdots + \frac{\partial \Phi}{\partial x_{r-2}^i} x_{r-1}^i - \frac{\partial \Phi}{\partial x_{r-1}^i} F^i, \tag{19}$$

$\Phi \, (x_{r-1}, \cdots, x_1, x, t)$ being any function of the variables in (17), we can write the above congruences in the form

$$d\omega_{s-1}^i \equiv da_k^i \, A_j^k \, \omega_{s-1}^j + a_k^i \Big\{ - dx_{s-1}^k \, dt - a_{s-1\,1,\,j}^k \, dx_{s-2}^j \, dt \tag{20}$$
$$- \left(\frac{da_{s-1\,1,\,j}^k}{dt} + a_{s-1\,2,\,j}^k \right) dx_{s-3}^j \, dt - \cdots$$
$$- \left(\frac{da_{s-1\,q-1,\,j}^k}{dt} + a_{s-1\,q,\,j}^k \right) dx_{s-q-1}^j \, dt \Big\} + \oplus,$$

where the symbol $\oplus$ denotes a sum of terms of the form $\omega_\alpha^i \, \omega_\beta^k$.

From (5) we get

$$\begin{cases} A_i^k \, \omega_{s-q}^i \, dt \equiv dx_{s-q-1}^k \, dt, \\ A_i^k \, \omega_{s-q+1}^i \, dt \equiv dx_{s-q}^k \, dt + a_{s-q+1\,1,\,j}^k \, dx_{s-q-1}^j \, dt, \\ \cdots\cdots\cdots\cdots \\ A_i^k \, \omega_s^i \, dt \equiv dx_{s-1}^k \, dt + a_{s\,1,\,j}^k \, dx_{s-2}^j \, dt + \cdots + a_{s\,q,\,j}^k \, dx_{s-q-1}^j \, dt. \end{cases} \tag{21}$$

These equations can be solved in terms of $dx^k_{s-q-1}\,dt, \cdots, dx^k_{s-1}dt$, giving relations of the form

$$
(22)\quad \begin{cases}
dx^k_{s-q-1}\,dt \equiv A^k_i\, \omega^i_{s-q}\, dt, \\
dx^k_{s-q}\,dt \equiv A^k_i\, \omega^i_{s-q+1}\,dt + b^k_{s-q\,1,\,j}\, A^j_i\, \omega^i_{s-q}\,dt, \\
\cdots\cdots\cdots\cdots \\
dx^k_{s-1}\,dt \equiv A^k_i\, \omega^i_s\, dt + b^k_{s-1\,1,\,j}\, A^j_i\, \omega^i_{s-1}\,dt + \cdots \\
\qquad + \left(-a^k_{s\,q,\,j} + b^k_{s-1\,q,\,j}\right) A^j_i\, \omega^i_{s-q}\,dt,
\end{cases}
$$

where the b's are known functions of the variables in (17), whose expressions can easily be found. By comparing (20) and (22), we see that the condition that the coefficient of $\omega^k_{s-q}\,dt$ in $d\omega^i_{s-1}$ be zero is given by

$$
(23)\qquad a^k_{s\,q,\,j} - b^k_{s-1\,q,j} - a^k_{s-1\,1,l}\, b^l_{s-2\,q-1,j} - \left(\frac{da^k_{s-1\,1,l}}{dt} + a^k_{s-1\,2,l}\right) b^l_{s-3\,q-2,j}
$$
$$
- \cdots - \left(\frac{da^k_{s-1\,q-1,j}}{dt} + a^k_{s-1\,q,j}\right) = 0.
$$

From these equations the $a^k_{s\,q,j}$ are completely determined.

By choosing the $a^k_{s\,q,j}$ in the above way, we can write the equations (12) in the form

$$
(24)\qquad d\omega^i_s = \theta^i_j\, \omega^j_s - \omega^i_{s+1}\, dt + \sum_{\substack{\alpha=1\\ \alpha\leq\beta}}^{s-1} \sum_{\beta=1}^{r} P^i_{s,\alpha\beta,jk}\, \omega^j_\alpha\, \omega^k_\beta, \quad s=2, 3, \cdots, r-1,
$$

where we may assume

$$
(25)\qquad P^i_{s,\alpha\alpha,jk} + P^i_{s,\alpha\alpha,kj} = o.
$$

In particular, we have, for $s=2$,

$$
(26)\qquad d\omega^i_2 = \theta^i_j\, \omega^j_2 - \omega^i_3\, dt + P^i_{2,11,jk}\, \omega^j_1\, \omega^k_1 + P^i_{2,12,jk}\, \omega^j_1\, \omega^k_2 + \cdots
$$
$$
+ P^i_{2,1r,jk}\, \omega^j_1\, \omega^k_r\, .
$$

The quantities $P^i_{2,12,jk}$ can be decompose into their symmetric and skew-symmetric parts, so that we may write

$$P^i_{2,12,jk} = \overset{*}{P^i_{2,12,jk}} + \overset{**}{P^i_{2,12,jk}},$$

where

$$\overset{*}{P^i_{2,12,jk}} + \overset{*}{P^i_{2,12,kj}} = o,$$

$$\overset{**}{P^i_{2,12,jk}} - \overset{**}{P^i_{2,12,kj}} = o.$$

Then we may write

$$d\omega^i_1 = \left(\theta^i_j + \overset{**}{P^i_{2,12,kj}} \omega^k_1 \right) \omega^j_1 - \omega^i_2\, dt,$$

$$d\omega^i_2 = \left(\theta^i_j + \overset{**}{P^i_{2,12,kj}} \omega^k_1 \right) \omega^j_2 - \omega^i_3\, dt + P^i_{2,11,jk}\, \omega^j_1\, \omega^k_1 + \overset{*}{P^i_{2,12,jk}}\, \omega^j_1\, \omega^k_2 + \cdots + P^i_{2,1r,jk}\, \omega^j_1\, \omega^k_r .$$

From these equations we see that there exists one, and only one, set of Pfaffian forms θ^i_j such that the equations (9) and (24) hold, with the conditions

$$(27) \qquad P^i_{2,12,jk} + P^i_{2,12,kj} = o.$$

The exact expression of θ^i_j is of the form

$$(11a) \qquad \theta^i_j = da^i_k A^k_j - a^i_k\, a^k_{21,l} A^l_j\, dt + a^i_m\, H^m_{pq} A^p_j A^q_k\, \omega^k_1,$$

where $H^m_{pq} = H^m_{qp}$ are functions of the variables in (17). By (10), these equations can also be written

$$(11b) \qquad \theta^i_j = - a^i_k\, dA^k_j - a^i_k\, a^k_{21,l} A^l_j\, dt + a^i_m\, H^m_{pq} A^p_j A^q_k\, \omega^k_1 .$$

We may also solve these equations in terms of da^i_k and dA^k_j, obtaining

$$(11c) \qquad \begin{cases} da^i_k = a^j_k\, \theta^i_j + a^i_l\, a^l_{21,k}\, dt - a^i_m\, H^m_{kq} A^q_l\, \omega^l_1 , \\ dA^k_j = - A^k_i\, \theta^i_j - a^k_{21,l} A^l_j\, dt + H^k_{pq} A^p_j A^q_l\, \omega^l_1 . \end{cases}$$

The θ^i_j being thus determined, we need still the expressions for $d\theta^i_j$. On applying the theorem of Poincaré to (9), we get

$$\left(d\theta^i_j - \theta^i_k\,\theta^k_j - \sum_{\alpha=1}^{r} P^i_{2,1\alpha,jk}\,\omega^k_\alpha\,dt\right)\omega^j_1 = 0. \tag{28}$$

This shows that if $d\theta^i_j - \theta^i_k\,\theta^k_j$ contains a term involving θ, it must be of the form $\theta^l_m\,\omega^p_1$. To see that it actually contains no term of this form, we apply the theorem of Poincaré to the equations (26). This gives

$$\left(d\theta^i_j - \theta^i_k\,\theta^k_j + dP^i_{2,12,kj}\,\omega^k_1 + P^k_{2,12,lj}\,\omega^l_1\,\theta^i_k - P^i_{2,12,kj}\,\omega^l_1\,\theta^k_l \right. \tag{29}$$
$$\left. - P^i_{2,12,lk}\,\omega^l_1\,\theta^k_j + \cdot\cdot\cdot\cdot\cdot\right)\omega^j_2 = o,$$

where the terms non-written are free from θ. Now, from the form of ω^i_2 we see easily that $P^i_{2,12,jk}$ is of the form

$$P^i_{2,12,jk} = a^i_m\, f^m_{pq}\, A^p_j\, A^q_k,$$

where f^m_{pq} is a function of the variables in (17). Hence

$$d\left(A^m_i\, P^i_{2,12,jk}\, a^j_p\, a^k_q\right) \equiv o \qquad (\mathrm{mod.}\ \omega, dt).$$

By developing this relation and taking account of (11), we obtain

$$dP^i_{2,12,kj} - P^l_{2,12,kj}\,\theta^i_l + P^i_{2,12,lj}\,\theta^l_k + P^i_{2,12,kl}\,\theta^l_j \equiv o \qquad (\mathrm{mod.}\ \omega, dt).$$

By comparing this with (28), (29) we conclude that $d\theta^i_j - \theta^i_k\,\theta^k_j$ does not contain θ. On the other hand, we have, from (28),

$$d\theta^i_j - \theta^i_k\,\theta^k_j - \sum_{\alpha=1}^{r} P^i_{2,1\alpha,jk}\,\omega^k_\alpha\,dt \equiv o \qquad (\mathrm{mod.}\ \omega_1).$$

Therefore $d\theta^i_j$ is necessarily of the form

$$(30)\qquad d\theta_j^i = \theta_k^i\,\theta_j^k + \sum_{\alpha=1}^{r} P^i_{2,1\alpha,jk}\,\omega_\alpha^k\,dt + T^i_{jk}\,\omega_1^k\,dt + \sum_{\alpha=1}^{r} R^i_{\alpha,jkl}\,\omega_\alpha^k\,\omega_1^l,$$

where, by (28), the quantities T^i_{jk} and $R^i_{\alpha,jkl}$ are connected by a set of relations to be given below.

We now write together the complete set of equations for the exterior derivatives of ω_γ^i, θ_j^i. By (9), (24), (14), (25), (27), (30), they are as follows:

$$(I)\quad \begin{cases} d\omega_1^i = \theta_j^i\,\omega_1^j - \omega_2^i\,dt, \\ d\omega_\gamma^i = \theta_j^i\,\omega_\gamma^j - \omega_{\gamma+1}^i\,dt + \sum\limits_{\substack{\alpha=1\\ \alpha\leq\beta}}^{\gamma-1}\sum\limits_{\beta=1}^{r} P^i_{\gamma,\alpha\beta,jk}\,\omega_\alpha^j\,\omega_\beta^k, \quad \gamma=2,3,\cdots r-1, \\ d\omega_r^i = \theta_j^i\,\omega_r^j + \sum\limits_{\substack{\alpha=1\\ \alpha\leq\beta}}^{r-1}\sum\limits_{\beta=1}^{r} P^i_{r,\alpha\beta,jk}\,\omega_\alpha^j\,\omega_\beta^k + \sum\limits_{\alpha=1}^{r-1} Q^i_{\alpha,k}\,\omega_\alpha^k\,dt, \\ d\theta_j^i = \theta_k^i\,\theta_j^k + \sum\limits_{\alpha=1}^{r} P^i_{2,1\alpha,jk}\,\omega_\alpha^k\,dt + T^i_{jk}\,\omega_1^k\,dt + \sum\limits_{\alpha=1}^{r} R^i_{\alpha,jkl}\,\omega_\alpha^k\,\omega_1^l, \end{cases}$$

with the following relations between the coefficients:

$$(II)\quad \begin{cases} P^i_{\gamma,\alpha\alpha,jk} + P^i_{\gamma,\alpha\alpha,kj} = 0, & \alpha = 1,\cdots, r-1, \\ & \gamma = 1,\cdots, r, \\ P^i_{2,12,jk} + P^i_{2,12,kj} = 0, & \\ T^i_{jk} - T^i_{kj} = 0, & \\ R^i_{\alpha,jkl} - R^i_{\alpha,lkj} = 0, & \alpha \neq 1, \\ R^i_{1,jkl} + R^i_{1,jlk} = 0, & \\ R^i_{1,jkl} + R^i_{1,klj} + R^i_{1,ljk} = 0. & \end{cases}$$

According to the theory of Cartan,(4) *the complete system of invariants of the differential system (1) is formed by the quantities P, Q, R, T and their covariant derivatives.* This gives a solution of the problem of equivalence.

We may also give a geometrical meaning to the fundamental equations (I). In fact, by the Pfaffian forms $\omega^i_\gamma, \omega^i_j$ we have defined in the space a generalized geometry, the definition being invariant under the group of transformations (2). Consider in particular the case when the given system is

$$\frac{d^r x^i}{dt^r} = o. \tag{31}$$

Then all the quantities P, Q, R, T in (I) vanish. Equations (1), together with

$$d\,(dt) = o, \tag{32}$$

are then the equations of structure of the group of transformations G:

$$\left\{\begin{array}{l} x^{i*} = a^i_k x^k + b^i_1\, t^{r-1} + b^i_2\, t^{r-2} + \cdots + b^i_{r-1}\, t + b^i_r, \\ t^* = t + A, \end{array}\right. \tag{33}$$

the Pfaffian forms ω, o being the relative components.(5) When the given system is of the general form (1), we may still interpret the ω, o as the components of an infinitesimal transformation of G. The space then becomes a non-holonomic space with the fundamental group G. The equations of the elements of the space are, from (I),

$$\omega^i_1 = \cdots = \omega^i_r = dt = 0.$$

They are formed by the elements of contact of the $(r-1)$st order and the parameter t. They constitute a space of $rn+1$ dimensions.

Summarizing the above results, we have shown that *given in the space R_n a system of differential equations (1) it is possible to define in the space a generalized geometry in the sense of Cartan, the definition being invariant under the group of transformations (2). The*

(4) E. Cartas, Les sousgroupes des groupes continus de transformations, *Annales de l'Ecole Normale Supérieure*, 1908.

(5) Cf. E. Cartan, La théorie des groupes continus et la géométrie différentielle traitée par la méthode du repère mobile, Paris 1937.

equations of structure of the space are given by (I), from which it follows that the fundamental group is the group G defined by (33) and the elements of the space are formed by the elements of contact of the $(r-1)$*st order* $x^i_{r-1}, \cdots, x^i$ *and the parameter* t.

§2. Relations of the Above Theory with the Absolute Differential Calculus

Strictly speaking, the geometry thus defined in R_n is not a generalized affine geometry (affine connection), as the group G is not an affine group. But as we have regarded t as an invariant, every transformation of G with t=const. will be an affine transformation. With the guidance of this remark we shall show that the above invariant theory can also be developed by the methods of the absolute differential calculus.

We begin by applying the theorem of Poincaré to the fundamental equations (I). By considering only those terms which contain θ, we get

$$dQ^i_{\alpha,k} + Q^i_{\alpha,l}\,\theta^l_k - Q^l_{\alpha,k}\,\theta^i_l \equiv o \quad (\text{mod.}\,\omega, dt),$$

$$dP^i_{\gamma,\alpha\beta,jk} + P^i_{\gamma,\alpha\beta,lk}\,\theta^l_j + P^i_{\gamma,\alpha\beta,jl}\,\theta^l_k - P^l_{\gamma,\alpha\beta,jk}\,\theta^i_l \equiv o \quad (\text{mod.}\,\omega, dt),$$

$$dT^i_{jk} + T^i_{lk}\,\theta^l_j + T^i_{jl}\,\theta^l_k - T^l_{jk}\,\theta^i_l \equiv o \quad (\text{mod.}\,\omega, dt),$$

$$dR^i_{\alpha,jkl} + R^i_{\alpha,mkl}\,\theta^m_j + R^i_{\alpha,jml}\,\theta^m_k + R^i_{\alpha,jkm}\,\theta^m_l - R^m_{\alpha,jkl}\,\theta^i_m \equiv o \quad (\text{mod.}\,\omega, dt).$$

By making use of (11a) and (11b), we see that these relations are equivalent to saying that the quantities Q, P, T, R are of the form

$$(34)\qquad \begin{cases} Q^i_k = a^i_m\, e^m_p\, A^p_k, \\ P^i_{jk} = a^i_m\, f^m_{pq}\, A^p_j A^q_k, \\ T^i_{jk} = a^i_m\, g^m_{pq}\, A^p_j A^q_k, \\ R^i_{jkl} = a^i_m\, h^m_{pqs} A^p_j A^q_k A^s_l, \end{cases}$$

where the e, f, g, h are functions of $x^i_{r-1}, \cdots, x^i, t$ and the Greek indices are omitted for simplicity.

As stated in §1, the complete system of invariants of the differential system (1) consists of the quantities P, Q, T, R and their covariant derivatives. By the covariant derivatives of an invariant $S(x, t, a)$ is meant the coefficients of the Pfaffian forms ω, dt, θ in the expression

$$(35) \qquad dS = S_{1,i}\, \omega^i_1 + \cdots + S_{r,i}\, \omega^i_r + S_o\, dt + S^l_k\, \theta^k_l .$$

Put, for example,

$$(36) \qquad dR^i_{\alpha,jkl} = \sum_{\beta=1}^{r} R^i_{\beta\alpha,jkl,m}\, \omega^m_\beta + R^i_{\alpha,jkl,o}\, dt - R^i_{\alpha,mkl}\, \theta^m_j$$
$$- R^i_{\alpha,jml}\, \theta^m_k - R^i_{\alpha,jkm}\, \theta^m_l + R^m_{\alpha,jkl}\, \theta^i_m .$$

The coefficients $R^i_{\beta\alpha,jkl,m}$, $R^i_{\alpha,jkl,0}$ are the covariant derivatives of $R^i_{\alpha,jkl}$ and are new invariants of the system. These new invariants are of forms like those in (34). This can easily be proved by making use of the fourth equation of (34) and the equations (11e). More generally, all the invariants obtained by the process of covariant differentiation from the quantities P, Q, T, R and their covariant derivatives are of forms like those in (34).

The auxiliary variables a^i_k have nothing to do with the elements of the generalized space, which depend on $x^i_{r-1}, \cdots, x^i$, t. We may interpret them as defining the reference (in French "repère") at the "point" $(x^i_{r-1}, \cdots, x^i, t)$. At each point of the generalized space we may attach a definite reference, called the natural reference, such that

$$a^i_k = \delta^i_k$$

and

$$\omega^i = dx^i - x^i_1\, dt, \text{ etc.}$$

By this definition, the system of natural references depends on the system of coordinates x^i in the space R_n. We choose the system of references such that in the coordinates x^i it is the system of natural references. When the coordinates undergo a transformation (2), we have

$$\bar{\omega}^i = \bar{a}^i_k\left(d\bar{x}^k - \bar{x}^k_1\, dt\right) = dx^i - x^i_1\, dt,$$

which gives

$$\bar{A}^i_k = \frac{\partial \bar{x}^i}{\partial x^k} \tag{37}$$

and

$$\bar{a}^i_k = \frac{\partial x^i}{\partial \bar{x}^k}\,. \tag{38}$$

The invariantive character of the quantities P, Q, T, R is then expressed by relations usually employed in the definition of tensors in absolute differential calculus. For instance, from the P^i_{jk} in (34) we have

$$\bar{P}^i_{jk} = P^i_{jk}\,,$$

or, with respect to the particular system of references chosen,

$$\bar{a}^i_m\, \bar{f}^m_{pq}\, \bar{A}^p_j\, \bar{A}^q_k = f^i_{jk}\,,$$

or, by (37) and (38),

$$\bar{f}^m_{pq} = \frac{\partial \bar{x}^m}{\partial x^i}\, f^i_{jk}\, \frac{\partial x^j}{\partial \bar{x}^p}\, \frac{\partial x^k}{\partial \bar{x}^q}\,. \tag{39}$$

Thus we see that our invariants are essentially the ordinary tensors in another form. The ordinary definition of tensors corresponds to a particular choice of the system of references made in terms of a definite system of coordinates. According to Cartan we call

$$P^i_{\gamma,\alpha\beta,jk}\,, \qquad \begin{array}{l} \gamma = 2, 3, \cdots, r \\ \alpha = 1, \cdots, \gamma-1,\ \beta = 1, \cdots, r \\ \qquad\qquad \alpha \leqq \beta \end{array}$$

$$Q^i_{\alpha,k} \qquad \alpha = 1, \cdots, r-1$$

the tensors of torsion and

$$T^i_{jk}, \qquad R^i_{\alpha,jkl}, \qquad \alpha = 1, \cdots, r$$

the tensors of curvature.

The covariant derivatives defined above can also be identified with those of the absolute differential calculus. In fact, we have, from (5),

$$\omega^i_\alpha = a^i_k \Big\{ dx^k_{\alpha-1} - x^k_\alpha\, dt + a^k_{\alpha 1,j}(dx^j_{\alpha-2} - x^j_{\alpha-1}\, dt) + \cdots$$
$$+ a^k_{\alpha\,\alpha-1,j}(dx^j - x^j_1\, dt) \Big\},$$

the a's within the braces being definite functions of $x_{r-1}, \cdots, x, t$. Consider, for definiteness, the functions P^i_{jk} defined by (34). By forming dP^i_{jk} and neglecting the terms in θ whose coefficients give essentially the same quantities P^i_{jk}, we get

$$dP^i_{jk} \equiv a^i_m \left(\frac{df^m_{pq}}{dt} + a^m_{21,l} f^l_{pq} - a^l_{21,p} f^m_{lq} - a^l_{21,q} f^m_{pl} \right) A^p_j A^q_k\, dt$$
$$+ a^i_m \left\{ \left(\frac{\partial f^m_{pq}}{\partial x^l} - H^m_{sl} f^s_{pq} + H^s_{pl} f^m_{sq} + H^s_{lq} f^m_{ps} \right)(dx^l - x^l_1\, dt) \right.$$
$$\left. + \frac{\partial f^m_{pq}}{\partial x^l_1}(dx^l_1 - x^l_2\, dt) + \cdots + \frac{\partial f^m_{pq}}{\partial x^l_{r-1}}(dx^l_{r-1} + F^l dt) \right\} A^p_j A^q_k, \qquad (\text{mod. } \theta).$$

By substituting the expressions of $dx^l - x^l_1\, dt, \cdots, dx^l_{r-1} + F^l\, dt$ in terms of the ω's, we get, as the coefficients of the ω's and of dt, the covariant derivatives of P^i_{jk}. Put

$$(40) \qquad dP^i_{jk} \equiv P^i_{1,jk,l}\, \omega^l_1 + P^i_{2,jk,l}\, \omega^l_2 + \cdots + P^i_{r,jk,l}\, \omega^l_r + P^i_{jk,o}\, dt \qquad (\text{mod. } \theta).$$

With respect to the system of natural references, so that $a^i_m = \delta^i_m$, the covariant derivatives of P^i_{jk} are defined by the relations

$$(41) \qquad \begin{cases} P^i_{jk,o} = \dfrac{dP^i_{jk}}{dt} + a^i_{21,l} P^l_{jk} - a^l_{21,j} P^i_{lk} - a^l_{21,k} P^i_{jl}, \\ P^i_{1,jk,l} + P^i_{2,jk,m}\, a^m_{21,l} + \cdots + P^i_{r,jk,m}\, a^m_{r\,r-1,l} \\ \qquad = \dfrac{\partial P^i_{jk}}{\partial x^l} - H^i_{sl} P^s_{jk} + H^s_{jl} P^i_{sk} + H^s_{lk} P^i_{js}, \\ P^i_{2,jk,l} + P^i_{3,jk,m}\, a^m_{31,l} + \cdots + P^i_{r,jk,m}\, a^m_{r\,r-2,l} = \dfrac{\partial P^i_{jk}}{\partial x^l_1}, \end{cases}$$

$$\begin{cases} \cdots\cdots\cdots\cdots\cdots\cdots \\ P^i_{r-1,jk,l} + P^i_{r,jk,m}\, a^m_{r1,l} = \dfrac{\partial P^i_{jk}}{\partial x^l_{r-2}}, \\ P^i_{r,jk,l} = \dfrac{\partial P^i_{jk}}{\partial x^l_{r-1}}, \end{cases}$$

The components $P^i_{jk,0}$, $P^i_{\alpha,jk,l}$ define $r+1$ new tensors, the former one being of the same order as P^i_{jk} and the latter ones of an order higher. Thus we see that *by the differential forms ω^l_α $r+1$ different processes of covariant differentiation can be defined.*

It may be mentioned that in the case of a vector u^i our covariant derivative $u^i_{,0}$ coincides with the differential operator Du^i of Kosambi.(6) But the other covariant derivatives are not the same as the so-called "Kosambian operators" $\underset{\gamma}{\nabla};u^i$.

§3. The Cases r=2 and r=3.

We shall give some explicit formulas for the tensors of curvature and of torsion and for the covariant differentiation in the special cases $r=2$ and $r=3$.

The case $r=2$ by which the given system of differential equations is of the second order has been thoroughly studied.(7) It will, however, be useful to give the results here. In this case, the equations of structure (I) become

$$(42)\quad \begin{cases} d\omega^i_1 = \omega^j_1\, \omega^i_j - \omega^i_2\, dt, \\ d\omega^i_2 = \omega^j_2\, \omega^i_j + P^i_{1,jk}\, \omega^j_1\, \omega^k_1 + P^i_{2,jk}\, \omega^j_1\, \omega^k_2 + Q^i_k\, \omega^k_1\, dt, \\ d\omega^i_j = \omega^k_j\, \omega^i_k + T^i_{jk}\, \omega^k_1\, dt + P^i_{2,jk}\, \omega^k_2\, dt + R^i_{1,jkl}\, \omega^k_1\, \omega^l_1 + R^i_{2,jkl}\, \omega^k_2\, \omega^l_1. \end{cases}$$

As discussed in the last section, we may simplify the calculation by taking the system of natural references, so that

(6) Cf. Bibliography No. 8, pp. 64-65.
(7) Cf. Bibliography Nos. 2 and 9.

$$
(43) \qquad \begin{cases} \omega_1^i = dx^i - x_1^i\, dt, \\[2ex] \omega_2^i = dx_1^i + F^i\, dt + \frac{1}{2} \dfrac{\partial F^i}{\partial x_1^j} (dx^j - x_1^j\, dt). \end{cases}
$$

By substituting these expressions in the first two equations of (42), we get

$$
(44) \qquad \begin{cases} \omega_j^i = -\dfrac{1}{2} \dfrac{\partial^2 F^i}{\partial x_1^j \partial x_1^k} (dx^k - x_1^k\, dt) - \dfrac{1}{2} \dfrac{\partial F^i}{\partial x_1^j}\, dt \\[2ex] P^i_{2,jk} = o, \\[2ex] P^i_{1,jk} = -\dfrac{1}{4} \dfrac{\partial^2 F^i}{\partial x_1^j \partial x^k} + \dfrac{1}{4} \dfrac{\partial^2 F^i}{\partial x_1^k \partial x^j} + \dfrac{1}{8} \dfrac{\partial^2 F^i}{\partial x_1^l \partial x_1^j} \dfrac{\partial F^l}{\partial x_1^k} \\[2ex] \qquad\qquad - \dfrac{1}{8} \dfrac{\partial^2 F^i}{\partial x_1^l \partial x_1^k} \dfrac{\partial F^l}{\partial x_1^j} \\[2ex] Q_j^i = \dfrac{\partial F^i}{\partial x^j} - \dfrac{1}{4} \dfrac{\partial F^i}{\partial x_1^k} \dfrac{\partial F^k}{\partial x_1^j} - \dfrac{1}{2} \dfrac{d}{dt} \left(\dfrac{\partial F^i}{\partial x_1^j} \right). \end{cases}
$$

These equations give the components of the tensor of torsion. For the calculation of the tensor of curvature, we may put

$$
(45) \qquad \begin{cases} dP^i_{1,jk} \equiv P^i_{1,jk,l}\, \omega_1^l + P^i_{1,jk/l}\, \omega_2^l + P^i_{1,jk/0}\, dt \qquad (\text{mod. } \theta), \\[2ex] dQ_k^i \equiv Q^i_{k,l}\, \omega_1^l + Q^i_{k1l}\, \omega_2^l + Q^i_{k10}\, dt \qquad (\text{mod. } \theta), \end{cases}
$$

and then apply the theorem of Poincaré to the second equation of (42). This gives

$$
(46) \qquad \begin{cases} T^i_{jk} = -Q^i_{k1j} - 2\, P^i_{1,jk}, \\[2ex] R^i_{1,jk} = -P^i_{1,kl1j}. \end{cases}
$$

By utilising (44_1), we have

$$
(47) \qquad R^i_{2,jkl} = -\frac{1}{2} \frac{\partial^3 F^i}{\partial x_1^j \partial x_1^k \partial x_1^l}.
$$

We may also give the formulas for covariant differentiation. Let

$$X^i = a^i_k\, \xi^k\,(x_{r-1}, \cdots, x, t)$$

be a vector and put

$$dX^i \equiv X^i_{,k}\,\omega^k_1 + X^i_{1k}\,\omega^k_2 + X^i_{1o}\,dt \qquad (\mathrm{mod.}\ \theta)$$

Equating the coefficients of $\omega^k_1, \omega^k_2, dt$ on both sides of the equation, we have, referred to the system of natural references,

$$(48)\qquad \left\{\begin{aligned} X^i_{,k} &= \frac{\partial X^i}{\partial x^k} - \frac{1}{2}\,\frac{\partial X^i}{\partial x^j_1}\,\frac{\partial F^j}{\partial x^k_1} + \frac{1}{2}\,\frac{\partial^2 F^i}{\partial x^j_1 \partial x^k_1}\,X^j, \\ X^i_{1k} &= \frac{\partial X^i}{\partial x^k_1}, \\ X^i_{1o} &= \frac{dX^i}{dt} + \frac{1}{2}\,\frac{\partial F^i}{\partial x^j_1}\,X^j. \end{aligned}\right.$$

Next, we consider the case of a system of differential equations of the third order ($r=3$). The equations of structure of the space are

$$(49)\qquad \left\{\begin{aligned} d\omega^i_1 &= \theta^i_j\,\omega^j_1 - \omega^i_2\,dt, \\ d\omega^i_2 &= \theta^i_j\,\omega^j_2 - \omega^i_3\,dt + \sum_{\alpha=1}^{3} P^i_{2,1\alpha,jk}\,\omega^j_1\,\omega^k_\alpha, \\ d\omega^i_3 &= \theta^i_j\,\omega^j_3 + \sum_{\alpha=1}^{3} P^i_{3,1\alpha,jk}\,\omega^j_1\,\omega^k_\alpha + P^i_{3,22,jk}\,\omega^j_2\,\omega^k_2 \\ &\qquad + P^i_{3,23,jk}\,\omega^j_2\,\omega^k_3 + Q^i_{1,k}\,\omega^k_1\,dt + Q^i_{2,k}\,\omega^k_2\,dt, \\ d\theta^i_j &= \theta^i_k\,\theta^k_j + \sum_{\alpha=1}^{3} P^i_{2,1\alpha,jk}\,\omega^k_\alpha\,dt + T^i_{jk}\,\omega^k_1\,dt \\ &\qquad + \sum_{\alpha=1}^{3} R^i_{\alpha,jkl}\,\omega^k_\alpha\,\omega^l_1. \end{aligned}\right.$$

By applying the general method described in §1 and employing the system of natural references, so that $a^i_k = \delta^i_k$, we get

$$
(50)\qquad \begin{cases}
\omega_1^i = dx^i - x_1^i\, dt, \\[4pt]
\omega_2^i = dx_1^i - x_2^i\, dt + \dfrac{1}{3}\dfrac{\partial F^i}{\partial x_2^j}(dx^j - x_1^j\, dt), \\[4pt]
\omega_3^i = dx_2^i + F^i\, dt + \dfrac{2}{3}\dfrac{\partial F^i}{\partial x_2^j}\,(dx_1^j - x_2^j\, dt) \\[4pt]
\qquad + \left\{ \dfrac{1}{3}\dfrac{d}{dt}\dfrac{\partial F^i}{\partial x_2^j} + \dfrac{1}{9}\dfrac{\partial F^i}{\partial x_2^l}\dfrac{\partial F^l}{\partial x_2^j} \right\}(dx^j - x_1^j\, dt).
\end{cases}
$$

These expressions, when substituted into the first two equations of (49), give

$$
(51)\qquad \begin{cases}
\omega_j^i = \lambda_{jh}^i\,(dx^h - x_1^h\, dt) - \dfrac{1}{3}\dfrac{\partial F^i}{\partial x_2^j}\, dt, \\[4pt]
P_{2,12,jk}^i = -\dfrac{1}{6}\dfrac{\partial^2 F^i}{\partial x_2^j \partial x_1^k} + \dfrac{1}{9}\dfrac{\partial^2 F^i}{\partial x_2^j \partial x_2^l}\dfrac{\partial F^l}{\partial x_2^k} \\[4pt]
\qquad + \dfrac{1}{6}\dfrac{\partial^2 F^i}{\partial x_2^k \partial x_1^j} - \dfrac{1}{9}\dfrac{\partial^2 F^i}{\partial x_2^k \partial x_2^l}\dfrac{\partial F^l}{\partial x_2^j}, \\[4pt]
P_{2,13,jk}^i = -\dfrac{1}{3}\dfrac{\partial^2 F^i}{\partial x_2^j \partial x_2^k},
\end{cases}
$$

where

$$
(52)\qquad \lambda_{jk}^i = -\frac{1}{6}\frac{\partial^2 F^i}{\partial x_2^j \partial x_1^k} + \frac{1}{9}\frac{\partial^2 F^i}{\partial x_2^j \partial x_2^l}\frac{\partial F^l}{\partial x_2^k} - \frac{1}{6}\frac{\partial^2 F^i}{\partial x_2^k \partial x_1^j} + \frac{1}{9}\frac{\partial^2 F^i}{\partial x_2^k \partial x_2^l}\frac{\partial F^l}{\partial x_2^j},
$$

while the expression for $P_{2,11,jh}^i$ is complicated and uninteresting.

By applying the theorem of Poincaré to the equations (49), we may show that the tensors

$$
P_{3,11,jk}^i,\ P_{3,13,jk}^i,\ P_{3,22,jk}^i,\ P_{3,23,jk}^i,\ T_{jk}^i,\ R_{1,jkl}^i,\ R_{3,jkl}^i
$$

can be expressed in terms of the tensors

$$
P_{2,11,jk}^i,\ P_{2,12,jk}^i,\ P_{2,13,jk}^i,\ P_{3,12,jk}^i,\ Q_{1,k}^i,\ Q_{2,k}^i,\ R_{2,jkl}^i
$$

and their covariant derivatives. Of the latter tensors we have

$$
(53)\qquad \left\{\begin{aligned}
Q^i_{1,j} &= \frac{\partial F^i}{\partial x^j} - \frac{1}{3}\frac{\partial F^i}{\partial x^l_1}\frac{\partial F^l}{\partial x^j_2} + \frac{2}{27}\frac{\partial F^i}{\partial x^l_2}\frac{\partial F^l}{\partial x^k_2}\frac{\partial F^k}{\partial x^j_2} \\
&\quad - \frac{2}{9}\frac{\partial F^i}{\partial x^l_2}\frac{d}{dt}\frac{\partial F^l}{\partial x^j_2} + \frac{2}{9}\frac{d}{dt}\left(\frac{\partial F^i}{\partial x^l_2}\right)\frac{\partial F^l}{\partial x_2{}^j} \\
&\quad - \frac{1}{3}\frac{d^2}{dt^2}\frac{\partial F^i}{\partial x^j_2}, \\
Q^i_{2,j} &= \frac{\partial F^i}{\partial x_1{}^j} - \frac{1}{3}\frac{\partial F^i}{\partial x^l_2}\frac{\partial F^l}{\partial x^j_2} - \frac{d}{dt}\frac{\partial F^i}{\partial x^j_2}, \\
R^i_{2,jkl} &= \frac{\partial \lambda^i_{jl}}{\partial x^k_1} - \frac{2}{3}\frac{\partial \lambda^i_{jl}}{\partial x^m_2}\frac{\partial F^m}{\partial x^k_2},
\end{aligned}\right.
$$

the expression for $P^i_{3,12;jk}$ being complicated.

We may also give the formulas for covariant differentiation. Let

$$X^i = a^i_k\, \xi^k\,(x_{r-1}, \cdots, x_1, x, t)$$

be a vector and put

$$(54)\qquad dX^i \equiv X^i_{1,k}\,\omega^k_1 + X^i_{2,k}\,\omega^k_2 + X^i_{3,k}\,\omega^k_3 + X^i_{10}\,dt \qquad (\text{mod. } \theta).$$

Referred to the system of natural references, we have

$$
(55)\qquad \left\{\begin{aligned}
X^i_{10} &= \frac{dX^i}{dt} + \frac{1}{3}\frac{\partial F^i}{\partial x^k_2}X^k, \\
X^i_{1,k} &= \frac{\partial X^i}{\partial x^k} - \frac{1}{3}\frac{\partial X^i}{\partial x^l_1}\frac{\partial F^l}{\partial x^k_2} + \frac{\partial X^i}{\partial x^l_2}\left(\frac{1}{9}\frac{\partial F^l}{\partial x^m_2}\frac{\partial F^m}{\partial x^k_2}\right. \\
&\quad \left. - \frac{1}{3}\frac{d}{dt}\frac{\partial F^l}{\partial x^k_2}\right) - \lambda^i_{kl}X^l, \\
X^i_{2,k} &= \frac{\partial X^i}{\partial x^k_1} - \frac{2}{3}\frac{\partial X^i}{\partial x^l_2}\frac{\partial F^l}{\partial x^k_2}, \\
X^i_{3,k} &= \frac{\partial X^i}{\partial x^k_2},
\end{aligned}\right.
$$

§4. The Differential System Studied under a Restricted Group of Transformations

We have studied the geometry of the differential system (1) under the group of transformations (2). All these properties are certainly invariant under the restricted group of transformations

$$
(56) \qquad \begin{cases} \bar{x}^i = \bar{x}^i(x^1, \cdots, x^n), \\ \bar{t} = t \end{cases} \qquad \left| \frac{\partial \bar{x}^i}{\partial x^k} \right| \neq o,
$$

But under the group (56) we have r more invariants which are the coefficients of dt in $\omega_1^i, \cdots, \omega_r^i$ and whose explicit expressions are

$$
(57) \qquad \begin{cases} v_1^i = a_k^i\, x_1^k, \\ v_2^i = a_k^i \left(x_2^k + a_{21,j}^k\, x_1^j \right), \\ \cdots\cdots\cdots \\ v_s^i = a_k^i \left(x_s^k + a_{s1,j}^k\, x_{s-1}^j + \cdots + a_{s\,s-1,j}^k\, x_1^j \right) \\ \cdots\cdots\cdots \\ v_r^i = a_k^i \left(-F^k + a_{r1,j}^k\, x_{r-1}^j + \cdots + a_{r\,r-1,j}^k\, x_1^j \right), \end{cases}
$$

where the $a_{s\alpha,j}^k$ are functions of $x_{r-1}^i, \cdots, x^i, t$ determined in §1. The equations in (6), together with the equality of the invariants in (57) and the corresponding invariants formed from the system (1′), are the necessary and sufficient conditions that the systems (1) and (1′) be equivalent under the grou of transformations (56). From the form of the invariants (57) we say that they are vectors in the generalized space. Hence the geometry of the differential system (1) under the group (56) is the geometry of the generalized space defined in §1 and the set of r vectors (57). This interpretation is not very satisfactory, as the vectors (57) have no relation with the generalized space defined above. In this section we want to show how in some particular cases it is possible to define a generalized geometry with a different fundamental group.

By the general theory the covariant derivatives of the vectors $v_1^i, \cdots, v_r^i$ give new invariants. Put

$$
(58) \qquad dv_\beta^i = v_{1\beta,j}^i\, \omega_1^j + \cdots + v_{r\beta,j}^i\, \omega_r^j + v_{\beta,o}^i\, dt + v_\beta^j\, \theta_j^i,
$$
$$
\beta = 1, 2, \cdots, r,
$$

we get the new invariants $v^i_{\alpha\beta,j}$, $v^i_{\beta,o}$. When $\beta=1$, it is easily proved that

$$v^i_{21,j} = \delta^i_j,\ v^i_{31,j} = \cdots = v^i_{r1,j} = o,\ v^i_{1,o} = v^i_2. \tag{59}$$

If we set

$$v^i_1 = \varepsilon^i_1,$$

where ε^i_1 are some properly chosen constant values, we have $dv^i_1 = d\varepsilon^i_1 = 0$ and

$$\omega^i_2 = - v^i_{11,j}\,\omega^j_1 - v^i_2\,dt - \varepsilon^j_1\,\theta^i_j. \tag{60}$$

Substituting (60) into the first equation of (I), we get

$$d\omega^i_1 = \theta^i_j\,\omega^j_1 + v^i_{11,j}\,\omega^j_1\,dt + \varepsilon^j_1\,\theta^i_j\,dt.$$

Put

$$\bar{\omega}^i_1 = \omega^i_1 + \varepsilon^i_1\,dt.$$

The above equation may be written

$$d\bar{\omega}^i_1 = \theta^i_j\,\bar{\omega}^j_1 + v^i_{11,j}\,\bar{\omega}^j_1\,dt. \tag{61}$$

From (60) the ω^i_2 are linear combinations of ω^j_1, dt, and θ^i_j. The equations giving the exterior derivatives of the other Pfaffian forms are of the form

$$\left\{\begin{aligned} d\bar{\omega}^i_1 &= \theta^i_j\,\bar{\omega}^j_1 + \bar{\Omega}^i_1, \\ d\omega^i_\gamma &= \theta^i_j\,\omega^j_\gamma - \omega^i_{\gamma+1}\,dt + \Omega^i_\gamma, \quad \gamma = 3, 4, \cdots, r-1, \\ d\omega^i_r &= \theta^i_j\,\omega^j_r + \Omega^i_r, \\ d\theta^i_j &= \theta^i_k\,\theta^k_j + \Theta^i_j, \end{aligned}\right. \tag{62}$$

where the $\bar{\Omega}^i_1$, Ω^i_γ $(\gamma = 3, \cdots, r)$, Θ^i_j are exterior quadratic forms. Thus our geometry may be regarded as a generalized geometry with the elements $(x^i_{r-1}, \cdots, x^i, t)$ and the fundamental group

$$
(63) \qquad \begin{cases} d\bar{\omega}^i_1 = \theta^i_j \,\bar{\omega}^j_1 \,, \text{[8]} \\ d\omega^i_\gamma = \theta^i_j \,\omega^j_\gamma - \omega^i_{\gamma+1}\, dt, \qquad \gamma = 3, 4, \cdots r-1 \\ d\omega^i_r = \theta^i_j \,\omega^j_r, \\ d\theta^i_j = \theta^i_h \,\theta^h_j \end{cases}
$$

such that there are given $r-1$ extra vectors $v^i_2, \cdots, v^i_r$ in the space. We see that by reducing the fundamental group to (63), the number of vectors in the space is reduced from r to $r-1$.

When $v^i_{32,j} = \delta^i_j$, we may put $v^i_2 = \varepsilon^i_2$= constants and express ω^i_3 in terms of $\omega^k_1, \omega^k_2, \omega^k_4, \cdots \omega^k_r, dt, \theta^k_j$. The process indicated above can still be applied and we can define a generalized geometry with a different fundamental group. But then the number of vectors in the space may not be reduced, as we get a new vector $v^i_{2,0}$ and we have still a system of $r-1$ vectors $v^i_{2,0}, v^i_3, \cdots, v^i_r$, which has no relation with the space. In general, by assigning to the components of these vectors and of their covariant derivatives fixed values, the a^i_k can be determined as functions of $x^i_{r-1}, \cdots, x^i, t$. The ω^i_α are then Pfaffian forms of $x^i_{r-1}, \cdots, x^i, t$ and we get a generalized geometry whose fundamental group is the group of translations

$$
(64) \qquad x^{i*} = x^i + a^i, \qquad t^* = t + A.
$$

In some particular cases, the definition of a generalized geometry with a fundamental group other than the group of translations (64) is possible. For example, when $r = 2$, and when the vector

$$
(65) \qquad v^i_2 = a^i_h \left(-F^h + \frac{1}{2} \frac{\partial F^h}{\partial x^j_1} x^j_1 \right) = o,
$$

the above discussion shows that we can define in the space an affine connection with the fundamental group

[8] It is to be noted that this group can not be represented by a group on the variables $x^1, \ldots, x^n$, t which is simply isomorphic to it.

$$
(66) \qquad \begin{aligned} x^{i*} &= a^i_k\, x^k + b^i, \\ t^* &= t + A. \end{aligned}
$$

This result we have also proved elsewhere.[9]

We may also consider the case $r=3$. By (50), the invariants (57) here become

$$
(67) \qquad \begin{cases} v^i_1 = a^i_k\, x^k_1, \\ v^i_2 = a^i_k \left(x^k_2 + \dfrac{1}{3} \dfrac{\partial F^k}{\partial x^j_2} x^j_1 \right), \\ v^i_3 = a^i_k \left(- F^k + \dfrac{2}{3} \dfrac{\partial F^k}{\partial x^j_2} x^j_2 + \dfrac{1}{3} \dfrac{d}{dt} \dfrac{\partial F^k}{\partial x^j_2} x^j_1 \right. \\ \qquad \left. + \dfrac{1}{9} \dfrac{\partial F^k}{\partial x^l_2} \dfrac{\partial F^l}{\partial x^j_2} x^j_1 \right). \end{cases}
$$

From the general discussion we can always define in the space a generalized geometry with the fundamental group (63) (for $r=3$) and with two extra vectors v^i_2, v^i_3. Put

$$
(68) \qquad dv^i_2 = v^i_{12,j}\, \omega^j_1 + v^i_{22,j}\, \omega^j_2 + v^i_{32,j}\, \omega^j_3 + v^i_{2,o}\, dt + v^j_2\, \theta^i_j.
$$

We find, by (55),

$$
(69) \qquad \begin{cases} v^i_{32,j} = \delta^i_j + \frac{1}{3}\, a^i_l \dfrac{\partial^2 F^l}{\partial x^k_2 \partial x^m_2} x^k_1\, A^m_j, \\ v^i_{2,o} = v^i_3. \end{cases}
$$

Suppose by the differential system under consideration the following conditions be verified:

$$
(70) \qquad \begin{cases} \dfrac{\partial^2 F^i}{\partial x^k_2 \partial x^j_2} x^k_1 = o, \\ - F^i + \dfrac{2}{3} \dfrac{\partial F^i}{\partial x^j_2} x^j_2 + \dfrac{1}{3} \dfrac{d}{dt} \dfrac{\partial F^i}{\partial x^j_2} x^j_1 + \dfrac{1}{9} \dfrac{\partial F^i}{\partial x^l_2} \dfrac{\partial F^l}{\partial x^j_2} x^j_1 = o. \end{cases}
$$

---

(9) Cf. Bibliography no. 9.

Then we get from (68), by setting the invariants v_2^i equal to some constants ε_2^i,

$$\omega_3^i = - v_{12,j}^i \omega_1^j - v_{22,j}^i \omega_2^j - \varepsilon_2^j \theta_j^i .$$

Substituting these expressions for ω_3^i into the second equation of (49), we can write it in the form

$$d(\omega_2^i + \varepsilon_2^i\, dt) = \theta_j^i (\omega_2^j + \varepsilon_2^j dt) + v_{12,j}^i \omega_1^j\, dt + v_{22,j}^i \omega_2^j\, dt$$
$$+ \sum_{\alpha=1}^{3} P_{2,1\alpha,jk}^i \omega_1^j \omega_\alpha^k .$$

Putting $\bar{\omega}_2^i = \omega_2^i + \varepsilon_2^i\, dt$, we can then write the equations of structure of the space in the form

$$(71) \qquad \begin{cases} d\bar{\omega}_1^i = \theta_j^i \bar{\omega}_1^j + \Omega_1^i , \\ d\bar{\omega}_2^i = \theta_j^i \bar{\omega}_2^j + \Omega_2^i , \\ d\omega_3^i = \theta_j^i \omega_3^j + \Omega_3^i , \\ d\theta_j^i = \theta_k^i \theta_j^k + \Theta_j^i , \end{cases}$$

in which $\Omega_1^i, \Omega_2^i, \Omega_3^i, \Theta_j^i$ are exterior quadratic forms in $\bar{\omega}_1^i, \bar{\omega}_2^i, \omega_3^i, dt$. This defines a space with the fundamental group

$$(72) \qquad \begin{cases} d\bar{\omega}_1^i = \theta_j^i \bar{\omega}_1^j , \\ d\bar{\omega}_2^i = \theta_j^i \bar{\omega}_2^j , \\ d\omega_3^i = \theta_j^i \omega_3^j , \\ d\theta_j^i = \theta_k^i \theta_j^k . \end{cases}$$

Therefore, *given in a space a system of differential equations of the third order by which the conditions (70) are satisfied, it is possible to define in the space a generalized geometry with the fundamental group (72).*

It is possible to obtain differential systems satisfying (70). For example, we may take

$$F^i = \alpha^i_j(x_1, x, t)\, x^j_2 + \beta^i(x_1, x, t). \tag{73}$$

Then the first equations in (70) are identically satisfied and the second equations give

$$\left\{\begin{aligned} \alpha^i_j &= \frac{\partial \alpha^i_k}{\partial x^j_1}\, x^k_1, \\ \beta^i &= \frac{1}{3}\frac{\partial \alpha^i_j}{\partial x^k}\, x^k_1\, x^j_1 + \frac{1}{3}\frac{\partial \alpha^i_j}{\partial t}\, x^j_1 + \frac{1}{9}\, \alpha^i_l\, \alpha^l_j\, x^j_1. \end{aligned}\right. \tag{74}$$

If $\alpha^i_j(x_1, x, t)$ is a set of functions satisfying (74_1) and the β^i are determined from (74_2), then the system of differential equations

$$\frac{d^3x^i}{dt^3} + \alpha^i_j\left(\frac{dx}{dt}, x, t\right)\frac{d^2x^j}{dt^2} + \beta^i\left(\frac{dx}{dt}, x, t\right) = 0 \tag{75}$$

gives rise to a geometry with the fundamental group (72).

§5. Some Particular Cases

If the given system of differential equations is

$$\frac{d^rx^i}{dt^r} = 0, \tag{31}$$

or reducible to this form by a transformation (2), all the tensors of torsion and tensors of curvature vanish. Conversely, *when the tensors of torsion and the tensors of curvature are zero, the given system can be reduced by a transformation of the group (2) to the form (31).*

Suppose we consider the case when the tensors of torsion are zero:

$$P^i_{\gamma,\alpha\beta,jk} = 0, \qquad Q^i_{\alpha,k} = 0.$$

By applying the theorem of Poincaré to the first equations of (I), we get

$$T^i_{jk}\, \omega^k_1\, \omega^j_\gamma\, dt = 0,$$

$$\sum_{\alpha=1}^{r} R^i_{\alpha,jkl}\, \omega^k_\alpha\, \omega^l_1\, \omega^j_\gamma = 0, \qquad \gamma = 1, 2, \cdots r.$$

The first equation gives

$$T^i_{jk} = 0$$

and the second equation gives, when $r \geqq 3$,

$$R^i_{\alpha,jkl} = 0.$$

Hence *when the tensors of torsion are zero, the tensors of curvature must be zero if* $r \geqq 3$. If $r=2$, the tensor $R^i_{2,jkl}$ is symmetric with respect to the indices j, k, l, but may not be zero.

Next consider the case when the tensors of curvature are zero:

$$P^i_{2,1\alpha,jk} = 0, \quad \alpha \neq 1,$$

$$P^i_{2,11,jk} + T^i_{jk} = 0, \quad R^i_{\alpha,jkl} = 0.$$

The application of the theorem of Poincaré to the equations of structure shows that the equations of structure must then be of the form

$$\text{(76)} \quad \begin{cases} d\omega^i_\gamma = \theta^i_j\,\omega^j_\gamma - \omega^i_{\gamma+1}\,dt, \qquad \gamma = 1, 2, \cdots, r-1, \\ d\omega^i_r = \theta^i_j\,\omega^j_r + \sum\limits_{\alpha=1}^{r-1} Q^i_{\alpha,k}\,\omega^k\;dt, \\ d\theta^i_j = \theta^i_k\;\theta^k_j, \end{cases}$$

and that we have

$$\text{(77)} \quad dQ^i_{\alpha,h} = Q^i_{\alpha,hj}\,\omega^j_\alpha + Q^i_{\alpha,h10}\,dt + Q^j_{\alpha,h}\,\theta^i_j - Q^i_{\alpha,j}\,\theta^j_h,$$

with

$$\text{(78)} \quad Q^i_{\alpha,kj} - Q^i_{\alpha,jk} = 0.$$

In this case, the equations of structure (76) may be written in a simple form. In fact, consider the system of differential equations

$$\text{(79)} \quad d\xi^i = \sum_{\alpha=1}^{r-1} Q^i_{\alpha,k}\,\omega^k_\alpha + \xi^j\,\theta^i_j + \xi^i_{10}\,dt.$$

To see whether this system possesses solutions $\xi^i_j\ \xi^i_{10}$ as functions of $x^k_{r-1}, \cdots x^k, t, a^i_k$, we form the equations obtained from (79) by exterior derivation. On taking account of the equations (77) and (78), we get

$$d\xi^i_{10}\,dt - \sum_{\alpha=1}^{r-1} Q^i_{\alpha,k10}\,\omega^k_\alpha\ dt - \sum_{\alpha=1}^{r-1} Q^i_{\alpha,k}\,\omega^k_{\alpha+1}\,dt - \xi^j_{10}\,\theta^i_j\,dt = 0. \tag{80}$$

According to the general theory of systems of differential equations(10) we easily see that the system (79) possesses solutions in the unknown functions $\xi^i_j\ \xi^i_{10}$. Moreover, from (79) we see that the ξ^i do not contain the variables x^k_{r-1} and that they are of the form

$$\xi^i = a^i_k\,\Phi^k(x_{r-2}, \cdots, x_1, x, t). \tag{81}$$

By making use of the functions ξ^i the equations of structure (76) may be written in the form

$$\begin{cases} d\omega^i_\gamma = \theta^i_j\,\omega^j_\gamma - \omega^i_{\gamma+1}\,dt, & \gamma = 1, 2, \cdots, r-1, \\ d\omega^i_r = \theta^i_j\,\omega^j_r + d\xi^i\,dt - \xi^j\,\theta^i_j\,dt, & \\ d\theta^i_j = \theta^i_k\,\theta^k_j. & \end{cases} \tag{82}$$

It may be shown that *when the functions F^i do not contain the derivatives of the $(r-1)$st order x^k_{r-1} the space is of zero curvature.* Conversely, we want to show that *a system of differential equations by which the generalized space defined is of zero curvature can be reduced by a transformation of the group (2) to the form*

$$\frac{d^r x^i}{dt^r} + F^i\left(\frac{d^{r-2}x}{dt^{r-2}}, \cdots, \frac{dx}{dt}, x, t\right) = 0, \tag{83}$$

in which the functions F^i do not contain the derivatives $\dfrac{d^{r-1}x^k}{dt^{r-1}}$. In fact, put

(10) Cf., for example, E. Kähler, *Theorie der Systeme von Dicerentialgleichungen.*

$$\bar{\omega}_r^i = \omega_r^i - \xi^i \, dt.$$

We get from (82)

$$d\bar{\omega}_r^i = \theta_j^i \, \bar{\omega}_r^j .$$

This shows that the Pfaffian system

$$\omega_1^i = 0, \ \cdots, \ \omega_{r-1}^i = 0, \quad \bar{\omega}_r^i = 0 \tag{84}$$

or the system of differential equations

$$\frac{d^r x^i}{dt^r} + F^i - \Phi^i = 0 \tag{85}$$

can be reduced by a transformation of the group (2) to the form (31). As the variables $\bar{x}_{r-2}^i, \bar{x}_{r-3}^i, \cdots, \bar{x}_1^i, \bar{x}^i, \bar{t}$ are functions of $x_{r-2}^i, x_{r-3}^i, \cdots, x_1^i, x^i, t$ only, transformation which reduces (85) to the form (31) must reduce the system

$$\frac{d^r x^i}{dt^r} + F^i = 0$$

to a system of the form (83).

Lastly, in order that the given system of differential equations be reducible by a transformation of the group (56) to the form (31), it is not sufficient that the tensors of torsion and curvature be zero. A necessary and sufficient condition for this is: 1) the tensors of torsion and curvature be zero, 2) the invariant v_r^i in (57) be zero, 3) the covariant derivatives of the invariants v_α^i ($\alpha = 1, 2, \cdots, r-1$) be given by equations of the form

$$\left\{ \begin{aligned} & dv_\alpha^i = \omega_{\alpha+1}^i + v_{\alpha+1}^i \, dt + v_\alpha^j \, \theta_j^i , \qquad \alpha = 1, 2, \cdots, r-2, \\ & dv_{r-1}^i = \omega_r^i + v_{r-1}^j \, \theta_j^i . \end{aligned} \right. \tag{86}$$

Bibliography

1. D. D. Kosambi, Parallelism and path-spaces, *Mathematische Zeitschrift*, 37 (1933), 608-618.
2. E. Cartan, Observations sur le mémo précédent, ibid 619-622.
3. Takeo Ohkubo, Die Geometrie der Differentialgleichungen dritter Ordnung, *Japanese Journal of Mathematics*, 14 (1938), 59-66.
4. V. Seetharaman, Differential invariants for path-spaces of order 3. *Proc. Indian Acad. Sci.* Section A 5 (1937), 336-342.
5. D. D. Kosambi, System of differential equations of the second order, *Quarterly Journal of Mathematics* 6 (1935), 1-12.
6. D. D. Kosambi, Path-spaces of higher order, *Quarterly Journal of Mathematics* 7 (1936), 97-104.
7. A. Kawaguchi and H. Hombu, Die Geometrie des Systems der partiellen Differentialgleichungen höherer Ordnung, *Journal of the Faculty of Science*, Hokkaido Imperial University, 6 (1937), 21-62.
8. V. Seetharaman, Differential invariants for higher path-spaces, *Proc. of London Math. Soc.*, 45 (1938), 64-80.
9. Shiing-shen Chern, Sur la géométrie d'un système d'équations différentielles du second ordre, *Bulletin des Sciences Mathématiques* 63 (1939), 206-212.

The National Tsing Hua University Kunming.

March 15, 1939.

(Received 2, April, 1939).